SYMBOLIC DYNAMICS FOR THE N-CENTRE PROBLEM AT NEGATIVE ENERGIES

NICOLA SOAVE AND SUSANNA TERRACINI

Abstract. We consider the planar N-centre problem, with homogeneous potentials of degree $-\alpha < 0$, $\alpha \in [1, 2)$. We prove the existence of infinitely many collisions-free periodic solutions with negative and small energy, for any distribution of the centres inside a compact set. The proof is based upon topological, variational and geometric arguments. The existence result allows to characterize the associated dynamical system with a symbolic dynamics, where the symbols are the partitions of the N centres in two non-empty sets.

1. Introduction

The N-centre problem consists in the study of the motion of a test particle moving under the action of the gravitational force fields of N fixed heavy bodies (the centres of the problem). In this paper we deal with the more general case of a Newtonian-like potential with homogeneity degree $-\alpha < 0$, with $\alpha \in [1, 2)$. The motion equation is

$$\ddot{x}(t) = -\sum_{j=1}^{N} \frac{m_j}{|x(t) - c_j|^{\alpha+2}} \left(x(t) - c_j\right), \tag{1.1}$$

where $x = x(t)$ denotes the position of the particle at the instant $t \in \mathbb{R}$, and c_j $(j = 1, \dots, N)$ is the position of the j-th centre. Introduced the potential

$$V(x) = \sum_{j=1}^{N} \frac{m_j}{\alpha\, |x - c_j|^{\alpha}}, \qquad x \in \mathbb{R}^2 \setminus \{c_1, \dots, c_N\},$$

we can rewrite equation (1.1) as a Newton equation

$$\ddot{x}(t) = \nabla V(x(t)),$$

which possesses a hamiltonian structure, of Hamiltonian

$$\frac{1}{2}|p|^2 - V(q) = h(p, q).$$

This paper concerns the existence of infinitely many collision-free periodic solutions with *negative energies*, for the planar planar N-centre problem; as a by-product of our construction, we will prove the occurrence of symbolic dynamics. The presence of chaotic trajectories has been established for positive energies in [5] and [17], while for small negative energies only perturbations of the two-centre cases have been treated ([7] and [12]).

Here we consider the case of small negative energies $h < 0$, and an arbitrary (even infinite) number of centres arbitrarily located inside a compact subset of the plane. A major difficulty of the negative energy case is that the energy shells

$$\mathcal{U}_h := \left\{ (x, v) \in \mathbb{R}^2 \setminus \{c_1, \dots, c_N\} \times \mathbb{R}^2 : \frac{1}{2}|v|^2 - V(x) = h \right\}$$

have a non empty boundary where the Jacobi metric degenerates. We shall seek trajectories as (non minimal) geodesics for the Jacobi metric and we shall exploit a broken geodesics method. Our method allows the simultaneous treatise of singularity of degree $\alpha = 1$ (Coulombic) and degree $\alpha > 1$ and provides collision-free trajectories.

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To describe our main result, we need some notation. Let us consider the possible partitions of the set of centres $\{c_1, \ldots, c_N\}$ in two disjoint non-empty sets (non ordered:$\{\{c_1\}, \{c_2, \ldots, c_N\}\} \simeq \{\{c_2, \ldots, c_N\}, \{c_1\}\}$). There are exactly

$$\frac{1}{2}\left(\binom{N}{1} + \ldots + \binom{N}{N-1}\right) = \frac{1}{2}\left(\sum_{k=0}^{N}\binom{N}{k} - 2\right) = 2^{N-1} - 1$$

such partitions. Each partition will be labeled within the set of the labels

$$\mathcal{P} := \{P_j : j = 1, \ldots, 2^{N-1} - 1\}.$$

It is convenient to distinguish those partitions which separate a single c_j from the others:

$$Q_j := \{\{c_j\}, \{c_1, \ldots, c_N\} \setminus \{c_j\}\} \qquad j = 1, \ldots, N.$$

This special kind of partitions define a subset of labels

$$\mathcal{P}_1 := \{Q_j \in \mathcal{P} : j = 1, \ldots, N\} \subset \mathcal{P}.$$

1.1. **Periodic solutions.** To any finite sequence of n symbols we associate an n-periodic bi-infinite sequence. We define the *right shift* in $\mathcal{P}^n$ as

$$T_r \colon \mathcal{P}^n \to \mathcal{P}^n \colon (P_{j_1}, P_{j_2}, \ldots, P_{j_n}) \mapsto (P_{j_n}, P_{j_1}, \ldots, P_{j_{n-1}}),$$

and we say that $(P_{j_1}, \ldots, P_{j_n}) \in \mathcal{P}^n$ *is equivalent to* $(P'_{j_1}, \ldots, P'_{j_n}) \in \mathcal{P}^n$ if there exists $m \in \mathbb{N}$ such that

$$(P'_{j_1}, \ldots, P'_{j_n}) = T_r^m\left((P_{j_1}, \ldots, P_{j_n})\right).$$

Our first goal consists in proving the existence of infinitely many periodic solutions at negative energies:

Theorem 1.1. *Let $\alpha \in [1, 2)$, $c_1, \ldots, c_N \in \mathbb{R}^2$, $m_1, \ldots, m_N \in \mathbb{R}^+$. There exists $\bar{h} < 0$ such that for every $h \in (\bar{h}, 0)$, $n \in \mathbb{N}$ and $(P_{j_1}, \ldots, P_{j_n}) \in \mathcal{P}^n$ there exists a periodic solution $x_{(P_{j_1}, \ldots, P_{j_n})}$ of the N-centre problem* (1.1) *with energy h, which depends on $(P_{j_1}, \ldots, P_{j_n})$ in the following way. There exist $\bar{R}, \bar{\delta} > 0$ (independent of $(P_{j_1}, \ldots, P_{j_n})$) such that $x_{(P_{j_1}, \ldots, P_{j_n})}$ passes alternatively outside and inside $B_{\bar{R}}(0)$, and*

- *if in (t_1, t_2) the solution stays outside $B_{\bar{R}}(0)$ and $x_{(P_{j_1}, \ldots, P_{j_n})}(t_1), x_{(P_{j_1}, \ldots, P_{j_n})}(t_2) \in \partial B_{\bar{R}}(0)$, then*
$$|x_{(P_{j_1}, \ldots, P_{j_n})}(t_1) - x_{(P_{j_1}, \ldots, P_{j_n})}(t_2)| < \bar{\delta}.$$
- *in its k-th passage inside $B_{\bar{R}}(0)$, if $x_{(P_{j_1}, \ldots, P_{j_n})}$ does not collide in any centre, then it separates the centres according to the partition P_{j_k}.*

To be precise:

(i) *if $\alpha \in (1, 2)$ then $x_{(P_{j_1}, \ldots, P_{j_n})}$ is collisions-free.*

(ii) *if $\alpha = 1$ there are three possibilities:*

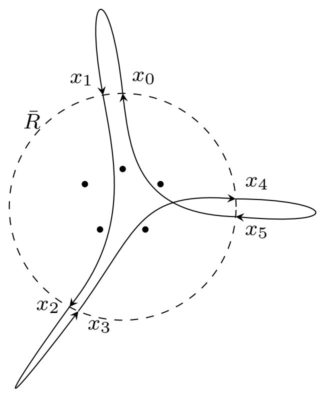
  a) *either $x_{(P_{j_1}, \ldots, P_{j_n})}$ is collisions-free.*

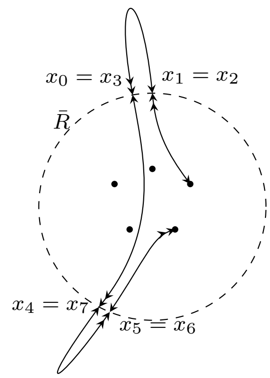
  b) *or $x_{(P_{j_1}, \ldots, P_{j_n})}$ has a collision with one centre c_j, covers a certain trajectory, rebounds against a second centre c_k (it may happen that $c_j = c_k$) and comes back along the same trajectory. Note that this is possible just if n is even and $(P_{j_1}, \ldots, P_{j_n})$ is equivalent to $(P'_{j_1}, \ldots, P'_{j_n})$ such that*
$$P'_{j_1} = Q_j \in \mathcal{P}_1, \quad P'_{j_{n/2+1}} = Q_k \in \mathcal{P}_1 \quad \text{and (if } n > 2\text{)}$$
$$P'_{j_n} = P'_{j_2},\ P'_{j_{n-1}} = P'_{j_3},\ \ldots,\ P'_{j_{n/2+2}} = P'_{j_{n/2}}.$$

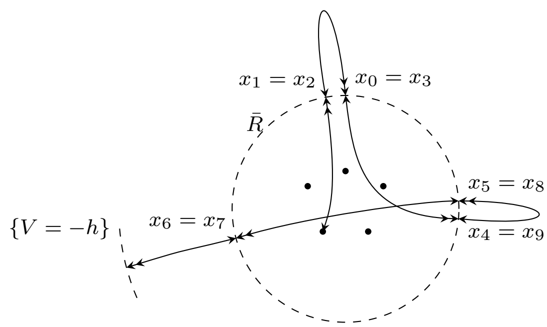
  c) *or else $x_{(P_{j_1}, \ldots, P_{j_n})}$ has a collision with one centre c_j, covers a certain trajectory, "rebounds" against the surface $\left\{x \in \mathbb{R}^2 : V(x) = -h\right\}$ with null velocity and comes back along the same trajectory. This is possible just if n is odd and $(P_{j_1}, \ldots, P_{j_n})$ is equivalent to $(P'_{j_1}, \ldots, P'_{j_n})$ such that*
$$P'_{j_1} = Q_j \in \mathcal{P}_1 \quad \text{and (if } n > 1\text{)}$$
$$P'_{j_n} = P'_{j_2},\ P'_{j_{n-1}} = P'_{j_3},\ \ldots,\ P'_{j_{(n+1)/2+1}} = P_{j_{(n+1)/2}}.$$

Of course, by varying both the number $n \in \mathbb{N}$ and the choice of the partitions $(P_{j_1}, \ldots, P_{j_n}) \in \mathcal{P}^n$, we find infinitely many periodic solutions. Let us also note that the symbol sequences of the collision solutions have a reflectional symmetry: by choosing non symmetric sequences we can rule out the occurrence of collisions.

The following pictures represent the case (i) or $(ii-a)$, $(ii-b)$, $(ii-c)$ respectively.







1.2. **Fixed ends problems.** To prove Theorem 1.1 we shall make use of a broken geodesics argument, finally leading to a finite dimensional reduction. A key step consists in solving fixed end problems having the desired topological characterization. This will yield to a constrained minimization for the Maupertuis' functional, the main difficulty being the possible interaction with the centres. Similar minimization problems in the presence of topological constraints have been recently treated in the literature concerning the variational approach to the N-body problem (see e.g. [3, 10, 11, 16]). We believe that the following intermediate result can be of independent interest.

Theorem 1.2. *Let $\alpha \in [1,2)$, $c_1, \ldots, c_N \in \mathbb{R}^2$, $m_1, \ldots, m_N \in \mathbb{R}^+$. There exist $\bar{h} < 0$ and $R > 0$ such that for every $h \in (\bar{h}, 0)$ and every pair of points $p_1, p_2 \in \partial B_R(0)$ and $P_j \in \mathcal{P}$, there exist $T > 0$ and a solution $y_{P_j}(\cdot\,; p_1, p_2)$ of the N-centre problem* (1.1) *at energy h such that $y_{P_j}(0) = p_1$, $y_{P_j}(T) = (p_2)$, $y_{P_j}(0,T) \subset B_R(0)$. Moreover:*

- (i) *if $\alpha \in (1,2)$ then y_{P_j} is collisions-free and self-intersections-free.*
- (ii) *if $\alpha = 1$ we have to distinguish among*
  - *a) $p_1 \neq p_2$; then y_{P_j} is collisions-free and self-intersections-free.*
  - *b) $p_1 = p_2$ and $P_j \in \mathcal{P} \setminus \mathcal{P}_1$; then y_{P_j} is collisions-free and self-intersections-free.*
  - *c) $p_1 = p_2$ and $P_j \in \mathcal{P}_1$; then y_{P_j} can be a collisions-free and self-intersections-free solution, or can be an ejection-collision solution, with a unique collision with c_j.*

Whenever it is collision free, then y_{P_j} separates the centres according to the partition P_j.

1.3. **Symbolic dynamics.** Let us consider the discrete metric space S (endowed with the trivial distance: $d_1(s_j, s_k) := \delta_{jk}\ \forall s_j, s_k \in S$, , where δ_{jk} is the Kronecker delta), and consider the bi-infinite sequences of elements of S:

$$S^{\mathbb{Z}} := \{(s_m)_{m\in\mathbb{Z}} : s_m \in S\ \forall m\}.$$

It is a metric space with respect to the distance

$$d((s_m),(t_m)) := \sum_{m\in\mathbb{Z}} \frac{1}{2^{|m|}} d_1(s_m, t_m), \qquad \forall (s_m),(t_m) \in S^{\mathbb{Z}}. \tag{1.2}$$

Of course, we can introduce a right shift in this space of sequences letting

$$T_r((s_m)) := (s_{m+1}) \qquad \forall (s_m) \in S^{\mathbb{Z}}.$$

Definition 1.1. Let Σ be a metric space, $\sigma : \Sigma \to \Sigma$ a continuous map, S a finite set. We say that the dynamical system (Σ, σ) has a *symbolic dynamics with set of symbols S* if there exist a σ-invariant subset Π of Σ and a continuous and surjective map $\pi : \Pi \to S^{\mathbb{Z}}$ such that the diagram

$$\begin{tikzcd} \Pi \arrow[r, "\sigma"] \arrow[d, "\pi"] & \Pi \arrow[d, "\pi"] \\ S^{\mathbb{Z}} \arrow[r, "T_r"] & S^{\mathbb{Z}} \end{tikzcd}$$

commutes, i.e. the restriction $\sigma|_\Pi$ is topologically semi-conjugate to the right shift in the metric space $(S^{\mathbb{Z}}, d)$ (d defined in (1.2)).

Let us write equation (1.1) as a first order autonomous Hamiltonian system:

$$\begin{cases} \dot{x}(t) = v(t) \\ \dot{v}(t) = \nabla V(x(t)) \end{cases} \Leftrightarrow \begin{pmatrix} \dot{x}(t) \\ \dot{v}(t) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} -\nabla V(x(t)) \\ v(t) \end{pmatrix}, \tag{1.3}$$

whit Hamiltonian function given by the energy.

Corollary 1.3. *Let $\bar{h}$ be introduced in Theorem 1.1, let $h \in (\bar{h}, 0)$. Then there exist a subset Π_h of the energy shell $\mathcal{U}_h$, a first return map $\mathfrak{R} : \Pi_h \to \Pi_h$, and a continuous and surjective map $\pi : \Pi_h \to \mathcal{P}^{\mathbb{Z}}$ such that the diagram*

$$\begin{tikzcd} \Pi_h \arrow[r, "\mathfrak{R}"] \arrow[d, "\pi"] & \Pi_h \arrow[d, "\pi"] \\ \mathcal{P}^{\mathbb{Z}} \arrow[r, "T_r"] & \mathcal{P}^{\mathbb{Z}}, \end{tikzcd}$$

commutes; namely for every $h \in (\bar{h}, 0)$ the dynamical system associated with the N-centre problem on the energy shells $\mathcal{U}^h$ has a symbolic dynamics, with set of symbols $\mathcal{P}$.

Although the planar N-centre problem appears as a simplified version of the restricted planar $(N+1)$-problem, in which the Coriolis' and the centrifugal forces are neglected, it is far away from being simple, except for the two-center problem, which is known to be integrable (see e.g. [30]). To give an idea of its complexity, we list below some remarkable results which are related with ours (our intent is not to give an exhaustive bibliography, and we refer the interested reader to the quoted works and the references therein): in [5], Bolotin proved the analytic non-integrability of the system for $N \geq 3$ on the energy shells for any $h > 0$. The question of analytic non-integrability has been faced also for the spatial problem by Knauf and Taimanov ([20]): they proved that an analytic integral which is independent with respect to the energy does not exist in case $n \geq 3$ and the energy is greater than some threshold h_{th}. The authors showed also the existence of smooth first integrals independent by the energy for both the planar and the spatial problem, with energy $h > 0$ and $h > h_{th}$ respectively. Another crucial reference, always for positive energies, is the work of Klein and Knauf [17], which provides an accurate description of the scattering for a wide class of problems having singular potentials. The spatial case was treated in [6]. As far as the negative energy case is concerned, the literature shows very few works, and the most remarkable results are obtained with fairly restrictive assumptions: in [7] Bolotin and Negrini proved the occurrence of chaotic dynamics for the 3-centre problem under the assumption of the third centre far away from the others two and a small absolute value of h; in [12] Dimare obtained a similar result for $h < 0$, $|h|$ small enough, in the case when one centre has small mass with respect to the others. In both papers the problem is approached in a perturbative setting.

1.4. **Plan of the paper.** Let us fix $\alpha \in [1, 2)$, $c_1, \ldots, c_N \in \mathbb{R}^2$, $m_1, \ldots, m_N > 0$. In section 2.1 we show equivalence of the fixed energy problem for (1.1) with small negative energies with the similar problem where the energy is normalized to -1 and the new centres lie inside a ball of radius ϵ. Hence we will deal with a rescaled potential

$$V_\epsilon(y) = \sum_{k=1}^{N} \frac{m_k}{\alpha |y - c'_k|^\alpha}, \qquad \max_{1 \leq k \leq N} |c'_k| < \epsilon.$$

The advantage of the reformulation is that, if ϵ is chosen sufficiently small, outside a ball of radius $R > \epsilon > 0$, the new problem is a small perturbation of the Kepler's problem with homogeneity degree $-\alpha < 0$ (we will call it "α-Kepler's problem"). This consideration leads to the search periodic solutions to

$$\begin{cases} \ddot{y}(t) = \nabla V_\epsilon(y(t)), \\ \frac{1}{2}|\dot{y}(t)|^2 - V_\epsilon(y(t)) = -1, \qquad \epsilon \in (0, \bar{\epsilon}) \end{cases} \tag{1.4}$$

dividing the investigation inside/outside the ball of radius $R > \bar{\epsilon} > 0$. In section 3 we will find arcs of solutions to (1.4) lying in $\mathbb{R}^2 \setminus B_R(0)$ connecting any pair of points $(x_0, x_1) \in (\partial B_R(0))^2$, provided their distance is sufficiently small, via perturbative techniques. In section 4 we will study the dynamics inside the ball $B_R(0)$ and provide solutions of (1.4) which connect $x_1, x_2 \in \partial B_R(0)$ for every x_1, x_2. This will be made trough a variational approach under suitable topological constraints. Finally, in section 5, we will collect the previous results to obtain periodic solutions of (1.4) which pass alternatively outside and inside $B_R(0)$, using a broken geodesics argument, through a finite dimensional reduction. Then, using the results of section 2.1, we will obtain a periodic solution of the original problem. Once we proved the existence Theorem 1.1, we will focus on the symbolic dynamics, proving Corollary 1.3 in section 6.

1.5. **Notations.** We will often identify a function u with the its image $u([a,b]) \subset \mathbb{R}^2$, with some abuse of notation. Throughout the paper C will be a strictly positive constant which may refer to different quantities even in the same proof. Sometimes it will be necessary to define different constants $C_1, \ldots, C_m$; in this case we point out that constants defined in one proof are not defined outside that proof.
It is convenient to introduce the polar coordinates for a point $x \in \mathbb{R}^2$:

$$x = re^{i\theta}, \qquad r > 0 \text{ and } \theta \in \mathbb{R}.$$

The angle θ is counted in counterclockwise sense, and $\theta = 0$ if $x = (1,0)$. For every continuous function $x : I \subset \mathbb{R} \to \mathbb{R}^2 \setminus \{0\}$, there exist continuous functions $r : I \to \mathbb{R}^+$ and $\theta : I \to \mathbb{R}$ such that

$$x(t) = r(t)e^{i\theta(t)}.$$

Dealing with the angular momentum of a $\mathcal{C}^1$ function x, we will write

$$\mathfrak{C}_x(t) := |x(t) \wedge \dot{x}(t)| = |r^2(t)\dot{\theta}(t)|$$

We will use the notations $\|\cdot\|_{L^p([a,b])}$ for the $L_p\left([a,b], \mathbb{R}^2\right)$-norm and $\|\cdot\|_{H^1([a,b])}$ for the $H^1\left([a,b], \mathbb{R}^2\right)$-norm; when there will not be possibility of misunderstanding, we will briefly write $\|\cdot\|_p$ or $\|\cdot\|$, respectively. The symbol $\rightharpoonup$ will denote the weak convergence in H^1.

2. Preliminaries

Let us fix $\alpha \in [1,2)$, $c_1, \ldots, c_N \in \mathbb{R}^2$, $m_1, \ldots, m_N > 0$. and fix the origin in the center of mass. Here and in what follows $M := \sum_{j=1}^N m_j$. In this subsection we prove that solving (1.1) with energy $h < 0$ is equivalent to solve a rescaled N-centre problem on the energy level -1. In this perspective the quadratic mean of the centers will replace the energy as a parameter. To be precise, we state the following elementary result.

Proposition 2.1. *Let $x \in \mathcal{C}^2\left((a,b); \mathbb{R}^2\right)$ be a classical solution of* (1.1) *with energy $h < 0$. Then the function*

$$y(t) = (-h)^{\frac{1}{\alpha}}\, x\left((-h)^{-\frac{\alpha+2}{2\alpha}}\, t\right), \qquad t \in \left((-h)^{\frac{\alpha+2}{2\alpha}}\, a, (-h)^{\frac{\alpha+2}{2\alpha}}\, b\right) \tag{2.1}$$

is a solution of energy -1 of a N-centre problem with centres

$$c'_j = (-h)^{\frac{1}{\alpha}}\, c_j, \qquad j = 1, \ldots, N.$$

The converse holds true: let $y \in \mathcal{C}^2\left((a', b'), \mathbb{R}^2\right)$ be a classical solution of energy -1 of a N-centres problem, with centres c'_j. Let us set

$$c_j = (-h)^{-\frac{1}{\alpha}}\, c'_j, \qquad j = 1, \ldots, N.$$

Then

$$x(t) = (-h)^{-\frac{1}{\alpha}}\, y\left((-h)^{\frac{\alpha+2}{2\alpha}}\, t\right), \qquad t \in \left((-h)^{-\frac{\alpha+2}{2\alpha}}\, a', (-h)^{-\frac{\alpha+2}{2\alpha}}\, b'\right)$$

is a classical solution of (1.1) *with energy $h < 0$.*

Corollary 2.2. *For every $\epsilon > 0$ there exists $\zeta(\epsilon) > 0$ such that if $-\zeta(\epsilon) < h < 0$, then the centres c'_j of the equivalent problem lye in $B_\epsilon(0)$. The function ζ is strictly decreasing in ϵ.*

Proof. Given $\epsilon > 0$ we find
$$\zeta(\epsilon) = -\left(\frac{\epsilon}{\max_{1\leq j\leq N}|c_j|}\right)^{\alpha}.$$
□

Remark 2.3. Of course, periodic solutions of the problem (1.4) for every $\epsilon \in (0,\bar{\epsilon})$, will correspond, via Proposition 2.1, to periodic solutions of (1.1) of energy $h = \zeta(\epsilon)$ for every $h \in (-\zeta(\bar{\epsilon}), 0)$.

As said, if ϵ is chosen sufficiently small, outside a ball of radius $R > \epsilon > 0$ we can consider the new problem as a small perturbation of the α-Kepler's problem: indeed let us consider the total potential: V_ϵ; then, for $|y| \geq R > \epsilon$ and $\max_j |c_j'| < \epsilon$, we have the expansion (in $\mathcal{C}^1(\mathbb{R}^2 \setminus B_R(0))$):
$$\nabla V_\epsilon(y) = \sum_{j=1}^{N} \frac{m_j}{\left|y-c_j'\right|^{\alpha+2}}(y-c_j) = \nabla\left(\frac{M}{\alpha|y|^\alpha}\right) + \mathcal{W}_\epsilon(y) = \nabla\left(\frac{M}{\alpha|y|^\alpha}\right) + o(\epsilon),$$
where
$$\mathcal{W}_\epsilon(y) = \nabla\left(V_\epsilon(y) - \frac{M}{\alpha|y|^\alpha}\right) \Rightarrow \mathcal{W}_\epsilon(y) = \nabla W_\epsilon(y).$$

Remark 2.4. If y is a solution of $\ddot{y} = \nabla V_\epsilon(y)$ over an interval $I \subset \mathbb{R}$, there holds
$$V_\epsilon(y(t)) \geq 1 \qquad \forall t \in I,$$
so that to exploit the previous argument we have to check that, for every $\epsilon > 0$ sufficiently small, there exists $R > 0$ such that
$$B_\epsilon(0) \subset B_R(0) \subset \left\{y \in \mathbb{R}^2 : V_\epsilon(y) \geq 1\right\}.$$
Of course, we can find such and R independent of the choice of ϵ, if ϵ is small enough. Then, for $y \in B_R(0)$, and for every $j \in \{1,\ldots,N\}$
$$|y-c_j'| \leq R+\epsilon \Rightarrow V_\epsilon(y) \geq \frac{M}{\alpha(R+\epsilon)^\alpha},$$
which is strictly greater than 1 if and only if $R < \left(\frac{M}{\alpha}\right)^{\frac{1}{\alpha}} - \epsilon$. There exists $\epsilon_1 > 0$ such that
$$0 < \epsilon < \epsilon_1 \Rightarrow \left(\frac{M}{\alpha}\right)^{\frac{1}{\alpha}} - \epsilon > \epsilon.$$
This argument shows that for every $\epsilon \in (0,\epsilon_1)$ there exists $R > 0$ such that $\epsilon_1 < R < \left(\frac{M}{\alpha}\right)^{\frac{1}{\alpha}} - \epsilon_1$. Actually, we will make the further request
$$\epsilon < \frac{R}{2} < R < \left(\frac{M}{\alpha}\right)^{\frac{1}{\alpha}} - \epsilon,$$
which is satisfied for every $\epsilon \in (0, \epsilon_1/2)$.
For reasons which appear clear in the section 4, *it is convenient to choose R such that $\partial B_R(0)$ is the support of the circular solution of the α-Kepler's problem with energy -1*; let $y(t) = R\exp\{i\omega t\}$; we find R such that
$$\ddot{y}(t) = -M\frac{y(t)}{|y(t)|^{\alpha+2}} \Leftrightarrow R\omega^2 = \frac{M}{R^{\alpha+1}}. \tag{2.2}$$
The conservation of the angular momentum $\mathfrak{C}_y(t) = |y(t) \wedge \dot{y}(t)|$ gives
$$R^2\omega = R\sqrt{2\left(-1+\frac{M}{\alpha R^\alpha}\right)}. \tag{2.3}$$
Collecting (2.2) and (2.3) we obtain
$$R := \left(\frac{(2-\alpha)M}{2\alpha}\right)^{\frac{1}{\alpha}}. \tag{2.4}$$
This is consistent with the previous restriction on R, if ϵ_1 is sufficiently small (if this was not true, it is sufficient to replace ϵ_1 with a smaller quantity).

We end this section with an elementary but useful remark. Proposition 2.1 and Corollary 2.2 enable us to switch from solutions x of (1.1) with energy $h < 0$, $|h|$ sufficiently small, to solutions y of (1.4). In this correspondence the topological properties of the solutions with respect to the centres are obviously preserved.

3. Outer dynamics

We are going to use a perturbation argument in order to find particular solutions of problem (1.4) lying in $\mathbb{R}^2 \setminus B_R(0)$, connecting pairs of neighbouring points of $\partial B_R(0)$ with a *close to brake* arc. To be precise we will prove the following theorem.

Theorem 3.1. *There exist $\delta > 0$ and $\epsilon_2 > 0$ such that for every $\epsilon \in (0, \epsilon_2)$, for every $p_0, p_1 \in \partial B_R(0)$: $|p_1 - p_0| < 2\delta$, there exist $T > 0$ and a unique solution $y_{ext}(\cdot\,; p_0, p_1; \epsilon)$ of 1.4 such that $|y(t)| > R$, for $t \in (0, T)$ and $y(0) = p_0$, $y(T) = p_1$. Moreover, y depends in a $\mathcal{C}^1$ way on the endpoints p_0 and p_1.*

The proof requires some preliminary results. We start from our unperturbed problem

$$\begin{cases} \ddot{y}(t) = -M\frac{y(t)}{|y(t)|^{\alpha+2}} & t \in [0, T], \\ \frac{1}{2}|\dot{y}(t)|^2 - \frac{M}{\alpha|y(t)|^\alpha} = -1 & t \in [0, T], \\ |y(t)| > R & t \in (0, T). \end{cases} \tag{3.1}$$

Let us solve the Cauchy problem

$$\begin{cases} \ddot{y}(t) = -M\frac{y(t)}{|y(t)|^{\alpha+2}} \\ y(0) = p_0 = R\exp\{i\theta_0\}, \quad \dot{y}(0) = \sqrt{2\left(-1 + \frac{M}{\alpha R^\alpha}\right)}\left(\frac{p_0}{R}\right). \end{cases}$$

The trajectory returns at the point p_0 after a certain time $\bar{T} > 0$, having swept the portion of the rectilinear brake orbit starting from p_0 and lying in $\mathbb{R}^2 \setminus B_R(0)$. Our aim is to catch the behaviour of the solutions under small variations of the boundary conditions. Hence, we consider

$$\begin{cases} \ddot{y}(t) = -M\frac{y(t)}{|y(t)|^{\alpha+2}} \\ y(0) = p_0, & \dot{x}(0) = \dot{r}_0 e^{i\theta_0} + R\dot{\theta}_0 i e^{i\theta_0}, \end{cases} \tag{3.2}$$

where $\dot{r}_0$ is assigned as function of $\dot{\theta}_0$ by means of the energy integral:

$$\dot{r}_0 = \dot{r}_0(\dot{\theta}_0) = \sqrt{2\left(\frac{M}{\alpha R^\alpha} - 1\right) - R^2\dot{\theta}_0^2}.$$

We denote as $y(\cdot\,; \theta_0, \dot{\theta}_0)$ the solution of (3.2). For the brake orbit $y(\cdot\,; \theta_0, 0)$ it results

$$\theta(t; \theta_0, 0) \equiv \theta_0 \qquad \forall t \in [0, \bar{T}].$$

Let us fix $p_0 \in \partial B_R(0)$. We define

$$\begin{aligned} \psi :& \Theta \times I \to \mathbb{R}^2 \\ & (\dot{\theta}_0, T) \mapsto y(T; \theta_0, \dot{\theta}_0), \end{aligned}$$

where $\Theta \times I \subset S^1 \times \mathbb{R}$ is a neighbourhood of $(0, \bar{T})$ on which ψ is well defined (such a neighbourhood exists). We can assume

$$\max\left\{\sup_{(\dot{\theta}_0, T)\in\Theta\times I} 4|T\dot{\theta}_0|, \sup_{(\dot{\theta}_0, T)\in\Theta\times I}\left|\left(\frac{\alpha}{M}\right)^{\frac{2}{\alpha}} R^2 T\dot{\theta}_0\right|\right\} < \frac{\pi}{2}, \tag{3.3}$$

otherwise it is sufficient to replace $\Theta \times I$ with a smaller neighbourhood.

Lemma 3.2. *The Jacobian of ψ in $(0, \bar{T})$ is invertible.*

Proof. Since the α-Kepler's problem is invariant under rotations, it isn't restrictive suppose $\theta_0 = \pi/2$, so that $\exp\{i\theta_0\} = (0, 1) =: e_2$. The function $\psi \in \mathcal{C}^1(\Theta \times I)$ satisfies

$$\frac{\partial\psi}{\partial T}(0, \bar{T}) = \dot{y}(\bar{T}; p_0, 0) = -\sqrt{2\left(\frac{M}{\alpha R^\alpha} - 1\right)} e_2.$$

Hence the Jacobian matrix of ψ is invertible in $(0, \bar{T})$ if

$$\left\langle \frac{\partial\psi}{\partial\dot{\theta}_0}(0, \bar{T}), e_1 \right\rangle \neq 0,$$

where $e_1 := (1,0)$. By continuous dependence with respect to initial data we have, for every $(\dot\theta_0, T) \in \Theta \times I$, and for every $t \in [0,T]$,

$$r(t;\theta_0,\dot\theta_0) \geq \frac{R}{2}. \tag{3.4}$$

We use the conservation of the angular momentum: for every $t \in [0,T]$ there holds

$$\mathfrak{C}_y := \mathfrak{C}_y(t) = |\mathfrak{C}_y(0)|, \Leftrightarrow \mathfrak{C}_y = r^2(t)|\dot\theta(t)| = R^2|\dot\theta_0|.$$

Assume $\dot\theta_0 > 0$; one has

$$\theta(t;\theta_0,\dot\theta_0) = \frac{\pi}{2} + \int_0^t \frac{d\theta}{ds}(s)\,ds = \frac{\pi}{2} + \int_0^t \frac{R^2\dot\theta_0}{r^2(s)}\,ds.$$

If $(\dot\theta_0, T) \in \Theta \times I$, from (3.3), (3.4), and the fact that $r(s) \leq (M/\alpha)^{1/\alpha}$, it follows

$$\frac{\pi}{2} < \left(\frac{\alpha}{M}\right)^{\frac{2}{\alpha}} R^2 T\dot\theta_0 + \frac{\pi}{2} \leq \theta(T;\theta_0,\dot\theta_0) \leq 4T\dot\theta_0 + \frac{\pi}{2} < \pi.$$

The function $\cos(\cdot)$ being decreasing over $(\pi/2, \pi)$, we obtain

$$\begin{aligned}\left\langle \frac{\psi(\dot\theta_0,\bar T) - \psi(0,\bar T)}{\dot\theta_0}, e_1 \right\rangle &= \frac{r(\bar T;\theta_0,\dot\theta_0)\cos\left(\theta(\bar T,\theta_0,\dot\theta_0)\right)}{\dot\theta_0} \\ &\leq \frac{r(\bar T;\theta_0,\dot\theta_0)\cos\left(\left(\frac{\alpha}{M}\right)^{\frac{2}{\alpha}} R^2 T\dot\theta_0 + \frac{\pi}{2}\right)}{\dot\theta_0} = -r(\bar T;\theta_0,\dot\theta_0)\left(\frac{\alpha}{M}\right)^{\frac{2}{\alpha}} R^2 T + o(\dot\theta_0^2) < 0,\end{aligned}$$

for $\dot\theta_0 \to 0$. Passing to the limit for $\dot\theta_0 \to 0$ the inequality is preserved. Since the same argument works for $\dot\theta_0 < 0$, the thesis follows. □

The previous discussion has to be refined in order to include the variations of the potential due to the presence of the centres, which are now included in the ϵ-disk. Recall that we fixed $p_0 \in \partial B_R(0)$. We know that

$$\lim_{\epsilon\to 0^+} V_\epsilon(y) = \frac{M}{\alpha|y|^\alpha} \qquad \text{uniformly in } y \in \mathbb{R}^2 \setminus B_R(0).$$

So we define

$$\begin{aligned}\Psi :&\Theta \times I \times \left[0, \frac{\epsilon_1}{2}\right) \times \partial B_R(0) \to \mathbb{R}^2 \\ &(\dot\theta_0, T, \epsilon, p_1) \mapsto y(T;\theta_0,\dot\theta_0;\epsilon) - p_1,\end{aligned}$$

where $y(\cdot\,;\theta_0,\dot\theta_0;\epsilon)$ is the solution of

$$\begin{cases}\ddot y(t) = \nabla V_\epsilon(y(t)) \\ y(0) = p_0, \qquad\qquad \dot y(0) = \dot r_\epsilon e^{i\theta_0} + R\dot\theta_0 i e^{i\theta_0},\end{cases} \tag{3.5}$$

and

$$\dot r_\epsilon = \dot r_\epsilon(\dot\theta_0;\epsilon) = \sqrt{2\left(V_\epsilon(p_0) - 1\right) - R^2\dot\theta_0^2}.$$

Lemma 3.3. *There exist $\delta > 0$ and $0 < \epsilon_2 < \epsilon_1/2$ such that for every $\epsilon \in (0,\epsilon_2)$, for every $p_1 \in \partial B_R(0)$: $|p_1 - p_0| < 2\delta$, there exists a unique solution $y(\cdot\,;\theta_0,\dot\theta_0;\epsilon)$ of (3.5) defined in $[0,T]$ for a certain T and satisfying*

$$\begin{aligned}\frac{1}{2}|\dot y(t;\theta_0,\dot\theta_0;\epsilon)|^2 - V_\epsilon(y(t;\theta_0,\dot\theta_0;\epsilon)) = -1 \qquad t \in [0,T], \\ |y(t;\theta_0,\dot\theta_0;\epsilon)| > R \qquad t \in (0,T), \qquad y(T;\theta_0,\dot\theta_0;\epsilon) = p_1.\end{aligned} \tag{3.6}$$

Moreover, it is possible to choose δ and ϵ_2 independent on $p_0 \in \partial B_R(0)$.

Proof. We apply the implicit function theorem to the function Ψ, which is $\mathcal{C}^1$ in the variables $\dot\theta_0$ and T for the differentiable dependence of the solutions by time and initial data. There holds

$$\Psi(0,\bar T,0,p_0) = 0, \quad \frac{\partial\Psi}{\partial\dot\theta_0}\left(0,\bar T,0,p_0\right) = \frac{\partial\psi}{\partial\dot\theta_0}\left(0,\bar T\right), \quad \frac{\partial\Psi}{\partial T}\left(0,\bar T,0,p_0\right) = \frac{\partial\psi}{\partial T}\left(0,\bar T\right),$$

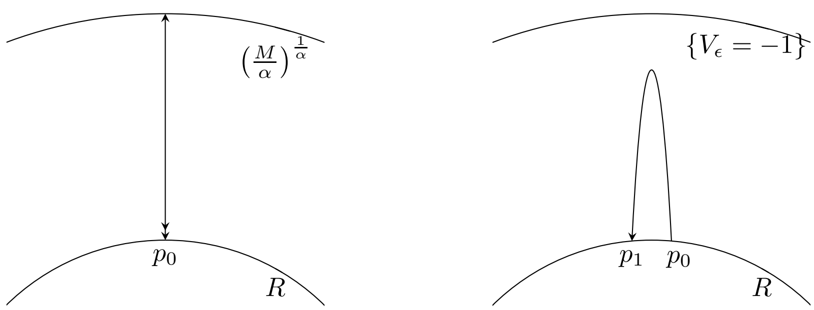
so that from Lemma 3.2 we deduce that the Jacobian matrix of Ψ with respect to $(\dot\theta_0, T)$ is invertible; hence the assumptions of the implicit function theorem are satisfied, and we can find a neighbourhood $\Theta' \times J \subset \Theta \times I$ of $(0, \bar T)$, a neighbourhood $[0, \epsilon_2) \times B_{2\delta}(p_0) \subset [0, \epsilon_1/2) \times \mathbb{R}^2$ of $(0, p_0)$, and a unique function $\eta : [0, \epsilon_2) \times B_{2\delta}(p_0) \to \Theta' \times J$ such that

$$\begin{aligned}
&1)\eta\,(0, p_0) = (0, \bar T),\\
&2)\Psi\,(\eta_1(\epsilon, p_1), \eta_2(\epsilon, p_1), \epsilon, p_1) = 0 \quad \text{for every } (\epsilon, p_1) \in [0, \epsilon_2) \times B_{2\delta}(p_0),\\
&3)\Psi(\dot\theta_0, T, \epsilon, p_1) = 0 \quad \text{with } (\dot\theta_0, T, \epsilon, p_1) \in \Theta' \times J \times [0, \epsilon_2) \times B_{2\delta}(p_0)\\
&\qquad \Rightarrow (\dot\theta_0, T) = \eta(\epsilon, p_1).
\end{aligned}$$

This means that, if we fix $\epsilon \in (0, \epsilon_2)$, for every $p_1 \in \partial B_R(0) \cap B_{2\delta}(p_0)$, we can find a solution $y(\cdot\,; \theta_0, \dot\theta_0; \epsilon)$ of (3.5). This solution has constant energy -1 because of the definition of $\dot r_\epsilon$; moreover, $y(T; \theta_0, \dot\theta_0; \epsilon) = p_1$. We remark that outside $B_R(0)$ the potential V_ϵ is a small perturbation of the α-Kepler's one, so that $|y(t; \theta_0, \dot\theta_0; \epsilon)| > R$ for every $t \in (0, T)$. It remains to prove that one can choose δ and ϵ_2 independent on x_0. This is a consequence of the proof of the implicit function theorem: the wideness of the neighbourhood of $(0, p_0)$ in $[0, \epsilon_1/2) \times \mathbb{R}^2$ in which we can guarantee the definition of the implicit function depends on the norm of

$$\left(J_{(\dot\theta_0, T)}\Psi(0, \bar T, 0, p_0)\right)^{-1},$$

and for every $p_0 \in \partial B_R(0)$ this matrix is the same up to rotations. $\square$



The picture represents the portion of the brake rectilinear solution for the α-Kepler's problem in comparison with a "perturbed" solution obtained for the potential V_ϵ via the implicit function theorem.

Theorem 3.1 is a straightforward consequence of this lemma. The solutions obtained are uniquely determined and depends in a smooth way on the ends p_0 and p_1.

4. Inner dynamics

In this section we are going to seek arcs of solutions of (1.4) connecting two points $p_1, p_2 \in \partial B_R(0)$ and lying inside the disk $B_R(0)$. We admit the case $p_1 = p_2$. Close to the center of the ball, the potential V_ϵ cannot be seen as a small perturbation of the α-Kepler's one, so that we are lead to use variational methods rather than perturbative techniques. The first step is to introduce a suitable functional whose critical points are weak solutions of (1.4); this will be made in subsection 4.1. Our trajectories will be local minimizers of the Maupertuis' functional or, equivalently, of the Jacobi length. More precisely, in subsection 4.2 we will determine weakly closed sets in which we will minimize the functional and we will state the main theorem of the section. It will be proved in 4.3 and 4.4; in the first one we will show that the direct method of the calculus of variations applies to provide weak solutions of (1.4), while in the latter one we will describe the behaviour of the solutions, proving in particular the absence of collisions in case $\alpha \in (1, 2)$. The case $\alpha = 1$ deserves a special analysis.
In what follows we will consider $\epsilon \in (0, \epsilon_1/2)$ fixed, and we will write c_j instead of c_j' to ease the notation. We

are going to seek solutions of

$$\begin{cases} \ddot{y}(t) = \nabla V_\epsilon(y(t)) & t \in [0,T], \\ \frac{1}{2}|\dot{y}(t)|^2 - V_\epsilon(y(t)) = -1 & t \in [0,T], \\ |y(t)| < R & t \in (0,T), \\ y(0) = p_1, \quad y(T) = p_2, \end{cases} \tag{4.1}$$

with $p_1, p_2 \in \partial B_R(0)$, and $T > 0$ to be determined.

4.1. **The Maupertuis' principle.** Dealing with a singular potential, we introduce the spaces on non-collision paths

$$\widehat{H}_{p_1 p_2}([a,b]) := \big\{ u \in H^1\left([a,b],\mathbb{R}^2\right) : u(a) = p_1,\ u(b) = p_2, \\ u(t) \neq c_j \text{ for every } t \in [a,b], \text{ for every } j \in \{1,\dots,N\} \ \big\}, \tag{4.2}$$

and

$$H_{p_1 p_2}([a,b]) := \left\{ u \in H^1\left([a,b],\mathbb{R}^2\right) : u(a) = p_1,\ u(b) = p_2 \right\}$$

(briefly $\widehat{H}$ and H). Let us note that, since $H^1\left([a,b],\mathbb{R}^2\right)$ is embedded in $\mathcal{C}\left([a,b],\mathbb{R}^2\right)$, the definitions are well posed. We point out that, defining

$$\mathfrak{Coll}_{p_1 p_2}([a,b]) := \big\{ u \in H^1\left([a,b],\mathbb{R}^2\right) : u(a) = p_1,\ u(b) = p_2, \\ \exists t \in [a,b] : u(t) = c_j \text{ for some } j \in \{1,\dots,N\} \ \big\},$$

the set of colliding functions in $H^1\left([a,b],\mathbb{R}^2\right)$ which connect p_1 with p_2, there holds

$$H_{p_1 p_2}([a,b]) = \widehat{H}_{p_1 p_2}([a,b]) \cup \mathfrak{Coll}_{p_1 p_2}([a,b])$$

and $H_{p_1 p_2}([a,b])$ is the closure of $\widehat{H}_{p_1 p_2}([a,b])$ in the weak topology of H^1. Let us define the *Maupertuis' functional*

$$M_h([a,b];\cdot) : H_{p_1 p_2}([a,b]) \to \mathbb{R} \cup \{+\infty\} \qquad M_h([a,b];u) = \frac{1}{2}\int_a^b |\dot{u}(t)|^2\,dt \int_a^b (V(u(t)) + h)\,dt.$$

We will often write M_h instead of $M_h([a,b];\cdot)$. If $M_h([a,b];u) > 0$ both its factors are strictly positive and it makes sense to set

$$\omega^2 := \frac{\int_a^b (V(u) + h)}{\frac{1}{2}\int_a^b |\dot{u}|^2}. \tag{4.3}$$

The Maupertuis' functional is differentiable over $\widehat{H}$ (seen as an affine space on H_0^1), and its critical points, suitably reparametrized, are solutions to our fixed energy problem (see [1]).

Theorem 4.1. *Let $u \in \widehat{H}_{p_1 p_2}([a,b])$ be a critical point of M_h at a positive level, i.e.*

$$dM_h([a,b];u)[v] = 0 \quad \forall v \in H_0^1\left([a,b],\mathbb{R}^2\right), \quad \textit{and} \quad M_h([a,b];u) > 0,$$

and let ω be given by (4.3)*. Then $x(t) := u(\omega t)$ is a classical solution of*

$$\begin{cases} \ddot{x}(t) = \nabla V(x(t)) & t \in \left[\frac{a}{\omega}, \frac{b}{\omega}\right], \\ \frac{1}{2}|\dot{x}(t)|^2 - V(x(t)) = h & t \in \left[\frac{a}{\omega}, \frac{b}{\omega}\right], \\ x\left(\frac{a}{\omega}\right) = p_1, & x\left(\frac{b}{\omega}\right) = p_2, \end{cases} \tag{4.4}$$

while u itself is a classical solution of

$$\begin{cases} \omega^2 \ddot{u}(t) = \nabla V(u(t)) & t \in [a,b], \\ \frac{1}{2}|\dot{u}(t)|^2 - \frac{V(u(t))}{\omega^2} = \frac{h}{\omega^2} & t \in [a,b], \\ u(a) = p_1, \quad u(b) = p_2. \end{cases} \tag{4.5}$$

Remark 4.2. The converse of Theorem 4.1 is also true: if $x \in \mathcal{C}^2\left([a',b'],\mathbb{R}^2\right)$ is a collisions-free solution of (4.4), setting $\omega = 1/(b'-a')$ and $u(t) := x(t/\omega)$, u is a classical solution of (4.5) defined in $[a'/(b'-a'), b'/(b'-a')] =: [a,b]$ and hence a critical point of $M_h([a,b];\cdot)$ at a strictly positive level. Also, the identity

$$\omega^2 = \frac{\int_a^b (V(u)+h)}{\int_a^b |\dot u|^2}.$$

is fulfilled.

In order to use variational methods it is worth working in H rather than in $\widehat{H}$, for $\widehat{H}$ isn't weakly closed. The disadvantage is that we will need some *ad hoc* argument to rule out the occurrence of collisions and to apply Theorem 4.1 and to obtain a classical solution of the motion equation. Nevertheless, although collision minimizers are not true critical points of the Maupertuis' functional on H, the following result allows to recover the conservation of the energy.

Lemma 4.3. *If $u \in H$ is a local minimizer of M_h, then*

$$\frac{1}{2}|\dot u(t)|^2 - \frac{V(u(t))}{\omega^2} = \frac{h}{\omega^2} \qquad a.e.\ t \in [a,b]$$

Remark 4.4. The lemma says that the energy is constant almost everywhere even if u has collisions. Of course, in this case u could be not of class $\mathcal{C}^1$. It is a classical result and it is a consequence of the extremality of u with respect to time reparametrization keeping the ends fixed: if $\varphi \in \mathcal{C}_c^\infty((a,b),\mathbb{R})$, setting $u_\lambda(t) := u(t+\lambda\varphi(t))$, there holds

$$\frac{d}{d\lambda} M_h(u_\lambda)\bigg|_{\lambda=0} = 0.$$

The Jacobi metric. The original Maupertuis' principle states that solutions of (4.4) are obtained, after a suitable reparametrization, as non-constant critical points of the functional

$$L_h(u) = L_h([a,b];u) := \int_a^b \sqrt{|\dot u(t)|^2 \left(V(u(t))+h\right)}\, dt,$$

which is defined on those $u \in H_{p_1p_2}([a,b])$ such that $V(u(t)) \geq -h$ for every $t \in [a,b]$. We define

$$H^* = H^*_{p_1,p_2}([a,b]) := \left\{u \in H : V(u(t)) > -h, |\dot u(t)| > 0 \text{ for every } t \in [a,b]\right\},$$

so that the domain of L_h is the closure of $H^*_{p_1p_2}([a,b])$ in the weak topology of H^1.
The functional $L_h(\gamma)$ has an important geometric meaning: it is the length of the curve parametrized by $\gamma \in H^*$ with respect to the Jacobi's metric:

$$g_{ij}(x) := (V(x)+h)\,\delta_{ij}, \quad \delta_{ij} = \begin{cases} 1 & i=j, \\ 0 & i \neq j \end{cases}, \quad i,j = 1,2.$$

The explicit expression of the reparametrization needed to pass from critical points of L_h to solution of (4.4) is the following. For $u \in H^1\left([a,b];\mathbb{R}^2\right)$ let us set

$$\Gamma_u := \left\{([a',b'],f) : f : [a',b'] \to [a,b], f \in \mathcal{C}^1([a',b'],\mathbb{R}) \text{ and increasing, such that } u \circ f \in H^1\left([a',b'],\mathbb{R}^2\right)\right\}.$$

Since L_h is a length, it is invariant under reparametrization: for every $u \in H^*_{p_1p_2}([a,b])$, for every $([a',b'],f) \in \Gamma_u$ it results

$$L_h([a,b];u) = L_h([a',b'];u\circ f).$$

It is well known that $u \in H^*$ is a critical point of L_h with respect to variations with compact support if and only if u solves the Euler-Lagrange equation

$$\frac{d}{dt}\left(\dot u \sqrt{\frac{V(u(t))+h}{|\dot u(t)|^2}}\right) - \frac{1}{2}\sqrt{\frac{|\dot u(t)|^2}{V(u(t))+h}}\nabla V(u(t)) = 0 \tag{4.6}$$

for almost every $t \in [a,b]$.

Theorem 4.5. *Let $u \in H^*_{p_1p_2}([a,b]) \cap \widehat{H}_{x_1,x_2}([a,b])$ be a non-constant critical point of $L_h([a,b];\cdot)$.*
Then there exist a reparametrization x of u which is a classical solution of (4.4) *in a certain interval* $[0, T/\sqrt{2}]$.

Proof. The function u is a collisions-free weak solution of (4.6), hence it is a strong solution. Define, for $t \in [a,b]$,

$$\theta(t) := \int_a^t \sqrt{\frac{|\dot{u}(z)|^2}{V(u(z))+h}}\, dz,$$

and set $T = \theta(b)$. It results $([0,T],\theta) \in \Gamma_u$ and for every $s \in [0,T]$ (we denote with "$'$" the differentiation with respect to the new parameter s)

$$\frac{dt}{ds}(s) = \left(\frac{d\theta}{dt}(t)\Big|_{t=t(s)} \right)^{-1} = \sqrt{\frac{V(u(t(s)))+h}{|\dot{u}(t(s))|^2}}.$$

With this change of variable, setting $y(s) = u(t(s))$, the (4.6) becomes

$$\frac{1}{t'(s)} \frac{d}{ds} \left(\frac{y'(s)}{t'(s)} t'(s) \right) - \frac{1}{2t'(s)} \nabla V(y(s)) = 0,$$

i.e.

$$y''(s) = \frac{1}{2} \nabla V(y(s)).$$

Setting $x(s) := y(\sqrt{2}s)$, x is a solution of the first equation in (4.4) in $[0, T/\sqrt{2}]$. As far as the second equation is concerned, for every $s \in [0, T/\sqrt{2}]$

$$|y'(s)|^2 = |\dot{u}(t(s)) t'(s)|^2 = V(y(s)) + h \Rightarrow \frac{1}{2}|x'(s)|^2 = V(x(s)) + h$$

which completes the proof. □

Relationship between L_h and M_h. It is convenient to establish a correspondence between minimizers of M_h at positive level and minimizers of L_h. This can be done through the simple inequality

$$L_h^2(u) = \left(\int_a^b \sqrt{|\dot{u}|^2 \left(V(u)+h\right)} \right)^2 \le \int_a^b |\dot{u}|^2 \int_a^b \left(V(u)+h\right) = 2M_h(u), \tag{4.7}$$

for every $u \in H^*$. The equality holds true if and only if there exists $\lambda \in \mathbb{R}$ such that for almost every $t \in [a,b]$

$$|\dot{u}(t)|^2 = \lambda \left(V(u(t)) + h \right).$$

Proposition 4.6. *Let $u \in H^* \cap \widehat{H}$ be a non-constant minimizer of M_h. Then u is a minimizer of L_h in $H^* \cap \widehat{H}$.*

Proof. Since u is a critical point of M_h in $\widehat{H}$ at a positive level, from Theorem 4.1 we know that

$$|\dot{u}(t)|^2 = \frac{2}{\omega^2} \left(V(u(t)) + h \right).$$

Hence there is equality in (4.7). If there existed $v \in H^* \cap \widehat{H}$ such that $L_h(v) < L_h(u)$, then we could reparametrize v to obtain a function (still denoted by v) satisfying

$$|\dot{v}(t)|^2 = \left(V(v(t)) + h \right)$$

(apply the argument in Theorem 4.5). So,

$$0 < 2M_h(v) = L_h^2(v) < L_h^2(u) = 2M_h(u),$$

a contradiction. □

Proposition 4.7. *If $u \in H^* \cap \widehat{H}$ is a non-constant minimizer of L_h then, up to reparametrization, u is a minimizer of M_h on $H^* \cap \widehat{H}$.*

Proof. We can assume from the beginning that there exists $\lambda \in \mathbb{R}$ such that for every $t \in [0,1]$

$$|\dot{u}(t)|^2 = \lambda \left(V(u(t)) + h \right).$$

Otherwise it is sufficient to perform a suitable reparametrization. Then there is equality in (4.7). Assume by contradiction that there existed $v \in H^* \cap \widehat{H}$ such that $M_h(v) < M_h(u)$. We can reparametrize v so that there is equality in (4.7). Therefore, we would deduce

$$L_h^2(v) = 2M_h(v) < 2M_h(u) = L_h^2(u),$$

a contradiction. □

Final comments. In this paper we will use both M_h and L_h. It is clear that the Maupertuis' functional M_h is easier to treat, so that it is convenient use it whenever possible. On the other hand the geometric meaning of the functional L_h will come useful. Indeed the couple set-metric given by

$$N = \left\{x \in \mathbb{R}^2 : V(x) > -h\right\}, \quad g_{ij}(x) = (V(x) + h)\,\delta_{ij}$$

(called the Hill's region) defines a Riemaniann manifold and we will take advantage of this structure, in spite to the degeneration of the metric on the boundary of the Hill region. More precisely, we will often make use of the following facts:

1) If $\gamma : [a,b] \to N$ is a piecewise differentiable curve, it is always possible to reparametrize it so that the length of the tangent vector
$$\sqrt{|\dot\gamma(t)|^2\,(V(\gamma(t)) + h)}$$
is a constant $C \in \mathbb{R}^+ \cup \{0\}$.
2) If a piecewise differentiable curve $\gamma : [a,b] \to N$, with parameter proportional to arc length, has length less or equal to the length of any other piecewise differentiable curve joining $\gamma(a)$ and $\gamma(b)$, then γ is a geodesic. In particular, γ is regular (recall that a geodesic is a curve satisfying the geodesics equation).
3) Let $p \in N$. We say that a subset $A \subset N$ is a *totally normal neighbourhood* of p if for every $p_1, p_2 \in \bar{A}$ there exists a unique minimizing geodesic γ joining p_1 and p_2. If this geodesic is contained in A, we say that A is a *strongly convex* neighbourhood.
For any $p \in N$ there exist a totally normal neighbourhood U of p. It is possible to choose U in such a way that U is strongly convex. If γ is the minimizing geodesic connecting p_1 and p_2 in U, γ depends smoothly on p_1 and p_2.

Furthermore we will strongly use the fact that, on contrarily to M_h, *the functional L_h is additive*. This is essential for the proof of the following Proposition.

Proposition 4.8. *Let $u \in H_{p_1p_2}([a,b])$ be a minimizer of $L_h([a,b];\cdot)$, let $[c,d] \subset [a,b]$. Then $u|_{[c,d]}$ is a minimizer of $L_h([c,d];\cdot)$ in $H_{u(c)u(d)}([c,d])$. Moreover, if u is a minimizer of $M_h([a,b];\cdot)$ in $H_{p_1p_2}([a,b])$, then, for any subinterval $[c,d] \subset [a,b]$, the restriction $u|_{[c,d]}$ is a minimizer of $M_h([c,d];\cdot)$ in $H_{u(c),u(d)}([c,d))$.*

4.2. **The existence theorem.** As said earlier, in order to find weak solutions of (4.1), we are going to minimize the Maupertuis' functional with some topological constraints. To this aim, the first step is to introduce suitable (possibly weakly closed) sets of functions. Let us fix $[a,b] \subset \mathbb{R}$ and $p_1, p_2 \in \partial B_R(0)$, $p_1 = R\exp\{i\theta_1\}$, $p_2 = R\exp\{i\theta_2\}$. The paths in $\widehat{H}$ can be classified according to their winding numbers with respect to each centre. This can be done by artificially closing it, in the following way:

$$\Gamma(t) := \begin{cases} \begin{cases} u(t) & t \in [a,b] \\ Re^{i(t-b+\theta_2)} & t \in (b, b+\theta_1+2\pi-\theta_2) \end{cases} & \text{if } \theta_1 < \theta_2 \\ u(t) \qquad t \in [a,b] & \text{if } \theta_1 = \theta_2 \\ \begin{cases} u(t) & t \in [a,b] \\ Re^{i(t-b+\theta_2)} & t \in (b, b+\theta_1-\theta_2) \end{cases} & \text{if } \theta_1 > \theta_2, \end{cases}$$

i.e. if $p_1 \neq p_2$ we close the path u with the arc of $\partial B_R(0)$ connecting p_2 and p_1 in counterclockwise sense. Then it is well defined the usual winding number

$$\operatorname{Ind}(u([a,b]), c_j) = \frac{1}{2\pi i}\int_\Gamma \frac{dz}{z - c_j}.$$

Given $l = (l_1, \dots, l_N) \in \mathbb{Z}^N$, a component of $\widehat{H}$ is of the form

$$\widehat{\mathfrak{H}}_l := \left\{u \in \widehat{H} : \operatorname{Ind}(u([a,b]), c_j) = l_j \quad \forall j = 1, \dots, N\right\}.$$

Remark 4.9. 1) In general $\widehat{\mathfrak{H}}_l$ may contain paths with self-intersections. Actually, $\widehat{\mathfrak{H}}_l$ contains self-intersections-free paths lying completely in $B_R(0)$ if and only if $l_j \in \{0,1\}$ for every j.
2) For every $l \in \mathbb{Z}^N$ the set $\widehat{\mathfrak{H}}_l$ is not weakly closed in H^1.

In the next subsection it will be useful to work on sets containing some self-intersections-free paths. For this reason we consider $l \in \mathbb{Z}_2^N$ instead of $l \in \mathbb{Z}^N$ and set

$$\widehat{H}_l := \left\{u \in \widehat{H} : \operatorname{Ind}(u([a,b]), c_j) \equiv l_j \mod 2 \quad \forall j = 1, \dots, N\right\},$$

namely we collect together the components having winding numbers having the same parity with respect to each centre. We also assume that

$$\exists j,k \in \{1,\dots,N\}\,,\ j \neq k,\ \text{such that } l_j \neq l_k \quad \text{mod } 2. \tag{4.8}$$

With this choice of l, if $u \in \widehat{H}_l$ then u has to pass through the ball $B_\epsilon(0)$ which contains the centres. In particular $u \in \widehat{H}_l$ cannot be constant even if $p_1 = p_2$, so that all the results stated in subsection 4.1 hold true even in this case. From now on, we will say that $l \in \mathbb{Z}_2^N$ is a *winding vector*.

In order to succeed in minimizing, we need to close those sets with respect to the weak H^1 topology. To this aim, we need to allow collisions with the centres. For $j \in \{1,\dots,N\}$, let us set

$$\begin{aligned}\mathfrak{Coll}_l^j := \{u \in H : \mathrm{Ind}\,(u([a,b]),c_k) \equiv l_k \quad \text{mod } 2\,, \forall k \in \{1,\dots,j-1,j+1,\dots,N\}\,,\\ \text{and there exists } t \in [a,b] : u(t) = c_j\}\,.\end{aligned}$$

A path $u \in \mathfrak{Coll}_l^j$ behaves as a path of $\widehat{H}_l$ with respect to c_k for $k \in \{1,\dots,j-1,j+1,\dots,N\}$ and collides in c_j at a certain instant. Analogously, for $j_1, j_2 \in \{1,\dots,N\}$ we define

$$\begin{aligned}\mathfrak{Coll}_l^{j_1,j_2} = \{u \in H : \mathrm{Ind}\,(u\,([a,b])\,,c_k) \equiv l_k \quad \text{mod } 2\,, \forall k \in \{1,\dots,N\} \setminus \{j_1,j_2\}\ ,\\ \text{and there are }\ t_1,t_2 \in [a,b] : u(t_1) = c_{j_1}, u(t_2) = c_{j_2}\}\,,\end{aligned}$$

the set of the paths behaving as paths of $\widehat{H}_l$ with respect to c_k for $k \in \{1,\dots N\} \setminus \{j_1,j_2\}$ and colliding in c_{j_1} and c_{j_2}; in the same way

$$\begin{aligned}&\mathfrak{Coll}_l^{j_1,j_2,j_3} := \dots,\\ &\vdots\\ &\mathfrak{Coll}_l^{1,\dots,N} = \mathfrak{Coll}^{1,\dots,N} := \{u \in H : u \text{ collides in each centre}\}\,.\end{aligned}$$

Finally, we name

$$\mathfrak{Coll}_l := \bigcup_{j=1}^N \mathfrak{Coll}_l^j \cup \bigcup_{1 \le j_1 < j_2 \le N} \mathfrak{Coll}_l^{j_1,j_2} \cup \dots \cup \mathfrak{Coll}_l^{1,\dots,N}.$$

Proposition 4.10. *The set*

$$H_l := \widehat{H}_l \cup \mathfrak{Coll}_l$$

is weakly closed in $H^1\left([a,b],\mathbb{R}^2\right)$.

Proof. Let $(u_n) \subset H_l$, $u_n \rightharpoonup u$ in H^1. Since the weak convergence in H^1 implies the uniform one, if u has a collision

$$(u_n) \subset H_l \Rightarrow u \in \mathfrak{Coll}_l.$$

If u is collisions-free, the uniform convergence implies the existence of $n_0 \in \mathbb{N}$ such that

$$u_n \in \widehat{H}_l \quad \forall n \ge n_0 \Rightarrow u \in \widehat{H}_l. \qquad \square$$

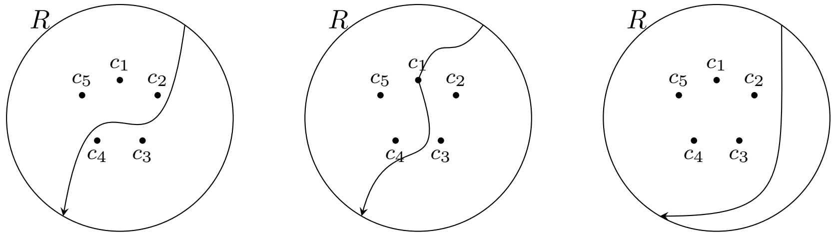
To complete the choice of suitable sets, it is convenient to add a further requirement: since we search functions lying in $B_R(0)$, let us set

$$\begin{aligned}\widehat{K}_l = \widehat{K}_l^{p_1 p_2}([a,b]) &:= \left\{u \in \widehat{H}_l : |u(t)| \le R\ \forall t \in [a,b]\right\}\\ K_l = K_l^{p_1 p_2}([a,b]) &:= \{u \in H_l : |u(t)| \le R\ \forall t \in [a,b]\}\,.\end{aligned}$$

Proposition 4.11. *The set* K_l *is weakly closed in* $H^1\left([a,b],\mathbb{R}^2\right)$.

Proof. K_l is a subset of the weakly closed set H_l, and it is stable under uniform convergence. $\square$

Some examples of paths: the first path is a collisions-free path with winding vector $(0,0,1,1,0)$; the second one is a collision path of K_l with $l = (0,1,1,0,0)$ or $l = (1,1,1,0,0)$; the third one is a path of K_l with $l = (0,0,0,0,0)$, which does not satisfy (4.8).



The main result of this section is the following theorem.

Theorem 4.12. *There exists $\epsilon_3 > 0$ such that for every $\epsilon \in (0, \epsilon_3)$, $p_1, p_2 \in \partial B_R(0)$ and $l \in \mathbb{Z}_2^N$ satisfying (4.8), there exist $T > 0$ and a solution $y_l(\cdot\,; p_1, p_2; \epsilon) \in K_l^{p_1 p_2}([0, T])$ of problem (4.1), which is a reparametrization of a local minimizer of the Maupertuis' functional in $K_l^{p_1 p_2}([0, 1])$. Moreover:*

- *(i) if $\alpha \in (1, 2)$ then y_l is collisions-free and self-intersections-free.*
- *(ii) if $\alpha = 1$ we have to distinguish among*
  - *a) $p_1 \neq p_2$; then y_l is collisions-free and self-intersections-free.*
  - *b) $p_1 = p_2$ and l is such that there exist $j_1, j_2, k_1, k_2 \in \{1, \ldots, N\}$:*

$$l_{j_1} = l_{j_2} \equiv 0 \mod 2 \quad l_{k_1} = l_{k_2} \equiv 1 \mod 2;$$

then y_l is collisions-free and self-intersections-free.

  - *c) $p_1 = p_2$ and l such that there exists $j \in \{1, \ldots, N\}$:*

$$l_1 = \cdots = l_{j-1} = l_{j+1} = \cdots = l_N \neq l_j \mod 2; \tag{4.9}$$

then y_l can be collisions-free and self-intersections-free or can be an ejection-collision solution, with a unique collision with one centre c_j.

Remark 4.13. The statement motivates us to say that an element $l \in \mathbb{Z}_2^N$ is a *collision winding vector* if it satisfies the (4.9). Let us also observe that the case (*ii-b*) makes sense just for $N \geq 4$.

The proof consists in an application of Theorem 4.1, at least for the cases (*i*), (*ii-a*), (*ii-b*). We will check that all its assumptions are satisfied in the next two subsections; in the latter one, we will also discuss the classification. We recall the

Definition 4.1. An *ejection-collision solution* of an equation

$$\ddot{x}(t) = \nabla V(x(t))$$

is a continuous function $x : I \subset \mathbb{R} \to \mathbb{R}^2$ such that

- there exists a collision set $T_c(x) \subset I$ such that for every $t^* \in T_c(x)$ there holds $x(t^*) = c_k$ for some $k = 1, \ldots, N$,
- the restriction $x|_{I \setminus T_c(x)}$ is a classical solution of

$$\ddot{x}(t) = \nabla V(x(t)),$$

- the energy is preserved trough collisions,
- at a collision instant, the trajectory is reflected:

$$x(t + t^*) = x(t^* - t) \qquad \forall t^* \in T_c(x), \forall t \in I \setminus T_c(x).$$

Before proceeding into the proof, we translate Theorem 4.12 in the language of partitions. To do this, we note that if $u \in \widehat{K}_l$ is self-intersections-free then it separates the centres in two different groups, which are determined by the particular choice of $l \in \mathbb{Z}_2^N$; namely, a self-intersections-free path in a class $\widehat{K}_l$ induces a partition of the centres in two sets. Since we are assuming (4.8), these sets are both non-empty. Hence it is well-defined an application $\mathcal{A} : \{l \in \mathbb{Z}_2^N : l \text{ satisfies (4.8)}\} \to \mathcal{P}$ which associate to a winding vector

$$l = (l_1, \ldots, l_N) \text{ with } \begin{cases} l_k \equiv 0 \mod 2 & k \in A_0 \subset \{1, \ldots, N\} \\ l_k \equiv 1 \mod 2 & k \in A_1 \subset \{1, \ldots, N\} \end{cases}$$

the partition

$$\mathcal{A}(l) := \{\{c_k : l_k \in A_0\}, \{c_k : l_k \in A_1\}\}.$$

This map is surjective but non injective, since for each couple $l, \widetilde{l} \in \mathbb{Z}_2^N$ such that

$$l_k \neq \widetilde{l}_k \mod 2 \qquad \forall k = 1, \dots, N,$$

then $\mathcal{A}(l) = \mathcal{A}(\widetilde{l})$.
Now it is natural to define

$$\widehat{K}_{P_j} = \widehat{K}_{P_j}^{p_1 p_2}([a,b]) := \left\{ u \in \widehat{K}_l : l \in \mathcal{A}^{-1}(P_j) \text{ and } u \text{ is self-intersections-free} \right\},$$
$$K_{P_j} = K_{P_j}^{p_1 p_2}([a,b]) := \left\{ u \in K_l : l \in \mathcal{A}^{-1}(P_j) \text{ and } u \text{ is self-intersections-free} \right\}.$$

They are respectively the set of the paths which connect p_1 and p_2 dividing the centres according to the partitions P_j, and its closure in the weak topology of H^1.
From Theorem 4.12, we obtain

Corollary 4.14. *Let ϵ_3 be introduced in Theorem 4.12. For every $\epsilon \in (0, \epsilon_3)$, $p_1, p_2 \in \partial B_R(0)$ and $P_j \in \mathcal{P}$, there exist $T > 0$ and a solution $y_{P_j}(\cdot\,; p_1, p_2; \epsilon) \in K_{P_j}^{p_1 p_2}([0,T])$ of problem* (4.1)*, which is a reparametrization of a local minimizer of the Maupertuis' functional M_{-1} in $K_{P_j}^{p_1 p_2}([0,1])$. Moreover:*

- (i) *if $\alpha \in (1,2)$ then y_{P_j} is collisions-free and self-intersections-free.*
- (ii) *if $\alpha = 1$ we have to distinguish among*
  - a) *$p_1 \neq p_2$; then y_{P_j} is collisions-free and self-intersections-free.*
  - b) *$p_1 = p_2$ and $P_j \in \mathcal{P} \setminus \mathcal{P}_1$; then y_{P_j} is collisions-free and self-intersections-free.*
  - c) *$p_1 = p_2$ and $P_j \in \mathcal{P}_1$; then y_{P_j} can be a collisions-free and self-intersections-free solution, or can be an ejection-collision solution, with a unique collision with c_j.*

4.3. **Minimization inside $B_R(0)$.** Let us fix $l \in \mathbb{Z}_2^N$ satisfying (4.8), and consider the restriction of the Maupertuis' functional M_{-1} to the set K_l. In this subsection we are going to provide weak solutions of (4.1) applying the direct method of the calculus of variations to M_{-1}. We fixed $[a,b] = [0,1]$, $p_1, p_2 \in \partial B_R(0)$; we will write M and L instead of M_{-1} and L_{-1}, respectively.

Remark 4.15. In the statement of Theorem 4.12 the value ϵ_3 depends neither on $p_1, p_2 \in \partial B_R(0)$, nor on $l \in \mathbb{Z}_2^N$, while here we fixed l before finding ϵ_3. Actually, once we found ϵ_3, we will see that it is independent on l.

The following statements are by now standard results and can be proved by routine applications of Poincaré inequality, Fatou's lemma and weak compactness arguments (see, for instance, [28, 26, 4]).

Lemma 4.16. *Assume* (4.8) *holds for $l \in \mathbb{Z}_2^N$. There exists a constant $C > 0$ such that*

$$M(u) \geq C > 0 \qquad \forall u \in K_l.$$

Proof. If $u \in K_l$, for every $j = 1, \dots, N$ and every $t \in [0,1]$, we have

$$|u(t) - c_j| \leq R + \epsilon \Rightarrow V_\epsilon(u(t)) \geq \frac{M}{\alpha (R+\epsilon)^\alpha}.$$

We required $\epsilon < R < \left(\frac{M}{\alpha}\right)^{\frac{1}{\alpha}} - \epsilon$, so there exists $\lambda_1 > 0$ such that

$$R = \left(\frac{M}{\alpha}\right)^{\frac{1}{\alpha}} - \epsilon - \lambda_1.$$

Thus for every $t \in [0,1]$

$$V_\epsilon(u(t)) - 1 \geq \frac{M}{\alpha\left(\left(\frac{M}{\alpha}\right)^{\frac{1}{\alpha}} - \lambda_1\right)} - 1 =: C > 0 \Rightarrow \int_0^1 (V_\epsilon(u) - 1) \geq C > 0.$$

Therefore the proof will be complete when we show the existence of $C > 0$ such that, for all $u \in K_l$, there holds

$$\|\dot{u}\|_2 \geq C. \tag{4.10}$$

Assume not: then there exists $(u_n) \subset K_l$ such that $\|\dot{u}_n\|_2 \to 0$. In particular $(\|\dot{u}_n\|_2) \subset \mathbb{R}$ is bounded. The sequence $(\|u_n\|_2)$ is bounded, too:

$$\int_0^1 |u_n(t)|^2 \, dt \leq R^2.$$

Then the sequence (u_n) is bounded in H^1, and this implies that up to subsequence $u_n \rightharpoonup v \in K_l$. Note that v must be a constant function. Thus $v(0) = p_1 = v(1) = p_2$. It is sufficient to note that, thanks to (4.8), v has to cross the ball $B_\epsilon(0)$ and therefore performs at least a distance $R - \epsilon$ and cannot be constant. □

Proposition 4.17. *Let $p_1, p_2 \in \partial B_R(0)$ (it is admissible $p_1 = p_2$). Let $l \in \mathbb{Z}_2^N$ satisfying (4.8). Then there exists a minimum of M on K_l at a positive level.*

Proof. Apply the direct method of the calculus of variations to the functional M defined on K_l: use Proposition 4.11 and Lemma 4.16, together with routine arguments of lower semi-continuity and coercivity. □

Let $l \in \mathbb{Z}_2^N$ satisfying (4.8) be fixed. If we show that the minimizer $u \in K_l$ is such that $|u(t)| < R$ for every $t \in (0,1)$, we can say that for every $\varphi \in \mathcal{C}_c^\infty([0,1], \mathbb{R}^2)$ there holds

$$\frac{d}{d\lambda} M(u + \lambda\varphi)\Big|_{\lambda=0} = 0,$$

so that u is a critical point of M at a positive level. In order to prove that $|u(t)| < R$, we follow the ideas in [2]; before proceeding, it is convenient recall a well known property of the solutions of the α-Kepler's problem.

Proposition 4.18. *Let $\alpha \in [1,2)$ and let $x : (a,b) \subset \mathbb{R} \to \mathbb{R}^2$ be a collision solution for the α-Kepler's problem with energy $h < 0$:*

$$\lim_{t \to b^-} x(t) = 0.$$

Then the angular momentum $\mathfrak{C}_x$ of x is 0.

Proof. In polar coordinates the energy is

$$\frac{1}{2}\dot{r}^2(t) + \frac{\mathfrak{C}_x^2}{2r^2(t)} - \frac{M}{\alpha r^\alpha(t)} = h \qquad \forall t \in (a,b).$$

In particular

$$h - \frac{\mathfrak{C}_x^2}{2r^2(t)} + \frac{M}{\alpha r^\alpha(t)} \geq 0 \qquad \forall t \in (a,b),$$

but if $\mathfrak{C}_x \neq 0$ then

$$\lim_{t \to b^-} h - \frac{\mathfrak{C}_x^2}{2r^2(t)} + \frac{M}{\alpha r^\alpha(t)} = -\infty,$$

a contradiction. Necessarily $\mathfrak{C}_x = 0$. □

Let us term

$$T_R(u) := \{t \in [0,1] : |u(t)| = R\}, \quad T_{R/2}^+(u) := \left\{t \in [0,1] : |u(t)| > \frac{R}{2}\right\}$$

A connected component of $T_R(u)$ is an interval (possibly a single point) $[t_1, t_2]$ with $t_1 \leq t_2$. The complement $T_{R/2}^+(u) \setminus T_R(u)$ is the union of a finite or countable number of open intervals.

Lemma 4.19. *A minimizer $u \in K_l$ of M has the following properties:*

(i) *If (a,b) is a connected component of $T_{R/2}^+(u) \setminus T_R(u)$, then $u|_{(a,b)}$ is of class $\mathcal{C}^2$ and is a solution of*

$$\omega^2 \ddot{u}(t) = \nabla V_\epsilon(u(t)), \quad \text{where} \quad \omega^2 := \frac{\int_0^1 (V_\epsilon(u) - 1)}{\frac{1}{2}\int_0^1 |\dot{u}|^2}.$$

(ii) *If $[t_1, t_2]$ is a connected component of $T_R(u)$, then $\theta|_{(t_1,t_2)}$ is $\mathcal{C}^2$, strictly monotone, and solves*

$$\ddot{\theta}(t) = \frac{1}{R\omega^2}\left\langle \nabla V_\epsilon(Re^{i\theta(t)}), ie^{i\theta(t)} \right\rangle. \tag{4.11}$$

(iii) *If $[t_1, t_2]$ is a connected component of $T_R(u)$, and (a,b) is a connected component of $T_{R/2}^+$ such that $[t_1, t_2] \subset (a,b)$, then one of the following situations occurs:*
  a) *$t_1 < t_2$ and $u \in \mathcal{C}^1((a,b))$,*
  b) *$t_1 = t_2$ and $u \in \mathcal{C}^1((a,b))$,*
  c) *$t_1 = t_2$ and $\dot{u}(t_1^-) \neq \dot{u}(t_1^+)$; in such a case u undergoes a radial reflection, i.e.*

$$\dot{r}(t_1^-) = -\dot{r}(t_1^+) \neq 0 \quad \text{and} \quad \dot{\theta}(t_1^-) = \dot{\theta}(t_1^+).$$

(*iv*) *There exist* $\epsilon_3 > 0$ *and* $\tau > 0$ *such that, if* $\epsilon \in (0, \epsilon_3)$*, for* t_3 *and* t_4 *satisfying*

$$|u(t_3)| = R, \quad |u(t_4)| = \frac{R}{2}, \quad \frac{R}{2} < |u(t)| < R \quad \forall t \in \begin{cases} (t_3, t_4) & \text{if } t_3 < t_4 \\ (t_4, t_3) & \text{if } t_3 > t_4 \end{cases},$$

there holds $|t_4 - t_3| \leq \tau$.

Proof. (*i*) It is a consequence of the minimality of u with respect to variations of u_n with compact support in (a, b). These variation are compatible with the constraint $\{x \in \mathbb{R}^2 : R/2 \leq |x| \leq R\}$.
(*ii*) For $t \in (t_1, t_2)$, the energy integral becomes

$$R^2 \dot{\theta}^2(t) = -\frac{2}{\omega^2} + \frac{2}{\omega^2} V_\epsilon \left(R e^{i\theta(t)} \right) \qquad \forall t \in [t_1, t_2]; \tag{4.12}$$

as a consequence $\theta \in \mathcal{C}^2((t_1, t_2))$. Since $V_\epsilon(R \exp\{i\theta\}) > 1$ for every $\theta \in [0, 2\pi]$, equation (4.12) implies that $\dot{\theta}(t) \neq 0$ for every $t \in (t_1, t_2)$. To get (4.11) it is sufficient to differentiate (4.12) with respect to t.
(*iii*) See the proof of Proposition 3.6 in [2].
(*iv*) In polar coordinates the energy integral reads

$$\frac{1}{2}\dot{r}^2(t) + \frac{\mathfrak{C}_u^2(t)}{2r^2(t)} - \frac{V_\epsilon\left(r(t)e^{i\theta(t)}\right)}{\omega^2} = -\frac{1}{\omega^2} \qquad \forall t \in [0, 1]. \tag{4.13}$$

It results

$$\frac{2}{\omega^2}\left(-1 + V_\epsilon\left(r(t)e^{i\theta(t)}\right)\right) - \frac{\mathfrak{C}_u^2(t)}{r^2(t)} \geq \frac{2}{\omega^2}\left(-1 + \frac{M}{\alpha(R+\epsilon)^\alpha}\right) + o(\epsilon);$$

The last equality is due to the fact that if we makes $\epsilon \to 0^+$, V_ϵ uniformly converges in the circular crown $R/2 \leq |x| \leq R$ to the potential of the Kepler's problem with homogeneity degree $-\alpha$. In particular, since u has to pass through the ball $B_\epsilon(0)$, which collapses in the origin, the angular momentum of u uniformly converges over the interval $[t_3, t_4]$ (or $[t_4, t_3]$) to 0 (see Proposition 4.18). From (4.13) we infer

$$|t_4 - t_3| \leq \int_{R/2}^{R} \frac{dr}{\sqrt{\frac{2}{\omega^2}\left(-1 + \frac{M}{\alpha(R+\epsilon)^\alpha}\right) + o(\epsilon)}}.$$

Since $-1 + \frac{M}{\alpha(R+\epsilon)^\alpha} > 0$ for every $\epsilon \in (0, \epsilon_1/2)$, there exists $0 < \epsilon_3 \leq \epsilon_1/2$ such that

$$\frac{2}{\omega^2}\left(-1 + \frac{M}{\alpha(R+\epsilon)^\alpha}\right) + o(\epsilon) \geq C > 0 \qquad \forall \epsilon \in (0, \epsilon_3),$$

and

$$|t_4 - t_3| \leq \frac{R}{2C} =: \tau.$$

□

Remark 4.20. From the proof of point (*iv*) it follows that ϵ_3 does not depend on $p_1, p_2 \in \partial B_R(0)$ or on $l \in \mathbb{Z}_2^N$, cf. Remark 4.15.

Lemma 4.21. *Let* $u \in K_l$ *be the minimizer found in Proposition 4.17, let* (a, b) *be a connected component of* $T^+_{R/2}$*. Then* $u \in \mathcal{C}^1((a, b))$*.*

Proof. If $u \in K_l$ is a minimizer of M, we show that situation $c)$ of point (*iii*) of previous lemma cannot occur. Recall that u is a minimizer also for L, which is the length with respect to the Jacobi's metric. If the situation $c)$ occurred, we could consider a totally normal neighbourhood U of the point $u(t_1)$ such that

$$\exists t_*, t_{**} \in (a, b) : u(t_*), u(t_{**}) \in u((a, b)) \cap \partial U.$$

If we connect $u(t_*)$ with $u(t_{**})$ with a minimizing arc for $L([t_*, t_{**}]; \cdot)$, we get a uniquely determined geodesic segment γ lying in U. In particular γ is regular, so that cannot coincide with $u|_{(t_*, t_{**})}$. Then, the curve

$$\widetilde{u}(t) := \begin{cases} u(t) & t \in [0, 1] \setminus [t_*, t_{**}] \\ \gamma(t) & t \in [t_*, t_{**}] \end{cases}$$

would be an element of K_l with $L(\gamma) < L(u)$, a contradiction. □

Proposition 4.22. *If $u \in K_l$ is the minimizer found in Proposition 4.17, then*

$$|u(t)| < R \quad \forall t \in (0,1).$$

Proof. Let $[t_1, t_2]$ be a connected component of $T_R(u)$, let (a,b) be a connected component of $T^+_{R/2}$ such that $[t_1, t_2] \subset (a,b)$. Let us consider $y(t) := u(\omega t)$. Since $y \in \mathcal{C}^1((a/\omega, b/\omega))$, it can lean against the circle $\{y \in \mathbb{R}^2 : |y| = R\}$ with tangential velocity, and for every $\nu > 0$ there exists $t_5 > t_2$ (or $t_5 < t_1$, and in this case the following inequality has to be changed in obvious way) such that

$$\left| y\left(\frac{t_5}{\omega}\right) - Re^{i\theta(t_2/\omega)} \right| < \nu \quad \text{and} \quad \left| \dot{y}\left(\frac{t_5}{\omega}\right) - R\dot{\theta}\left(\frac{t_2}{\omega}\right) ie^{i\theta(t_2/\omega)} \right| < \nu.$$

Thus

- R is the radius of the circular solution of energy -1 for the Kepler's problem with homogeneity degree $-\alpha$,
- outside $B_{R/2}(0)$, the N-centres problem can be seen as a small perturbation of the α-Kepler's one: $V_\epsilon(y) = \frac{M}{\alpha|y|^\alpha} + W_\epsilon(x)$.
- y is a solution of
$$\begin{cases} \ddot{y}(t) = \nabla V(y(t)) \\ y\left(\frac{t_5}{\omega}\right) \simeq Re^{i\theta(t_2/\omega)}, \quad \dot{y}\left(\frac{t_5}{\omega}\right) \simeq R\dot{\theta}\left(\frac{t_2}{\omega}\right) ie^{i\theta(t_2/\omega)}. \end{cases}$$
in an open neighbourhood of t_5/ω; these initial data are "more or less" the initial data of a circular solution.
- the theorem of continuous dependence of the solutions with respect to the vector field and the initial data holds true for our problem outside $B_{R/2}(0)$.

Therefore y cannot enter (or exit from) the ball $B_{R/2}(0)$ in a finite time, in contradiction with the choice of l and point (iv) of Lemma 4.19. □

4.4. **Classification of the minimizers.** So far, we obtained a set of extremals of the Maupertuis' functional M at positive levels. In order to obtain classical solutions to (4.1), we need to show that these minimizers are collisions-free. In case $\alpha = 1$ this fact isn't always true; however, we will be able to describe the behaviour of the collision-solutions, proving the classification in Theorem 4.12.

The proof is by contradiction and requires several steps. In what follows we consider $l \in \mathbb{Z}_2^N$ satisfying (4.8) and fixed. We assume that the minimizer $u \in K_l$ has at least one collision; developing a blow-up analysis at the collision, we will reach a contradiction in the case $\alpha \in (1,2)$; the case $\alpha = 1$ will be more difficult and will be treated separately by Levi-Civita regularization.

Step 1). We prove that *u has no self-intersections at points different from the centres and that the set of collision times of u*

$$T_c(u) := \{t \in [0,1] : u(t) = c_j \text{ for some } j \in \{1, \dots, N\}\}$$

is finite.

Since $M(u) < +\infty$, it follows immediately that $T_c(u)$ is a closed set of null measure. Hence $[0,1] \setminus T_c(u)$ is the union of a finite or countable number of open intervals. We recall that the energy of u is constant and equal to $-1/\omega^2$, see Lemma 4.3, where the value ω has been already defined in Lemma 4.19.

Lemma 4.23. *If the interval (a,b) is a connected component of $[0,1] \setminus T_c(u)$, then $u|_{(a,b)} \in \mathcal{C}^2((a,b), \mathbb{R}^2)$ and*

$$\omega^2 \ddot{u}(t) = \nabla V_\epsilon(u(t)) \qquad \forall t \in (a,b). \tag{4.14}$$

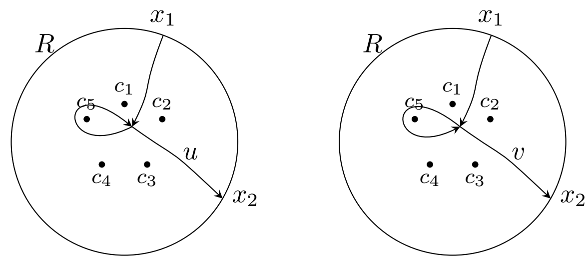
Proof. It is enough to repeat the proof of point (i) of Lemma 4.19. □

Proposition 4.24. *The minimizer u parametrizes a path without self-intersections at points different from the centres c_j $(j = 1, \dots, N)$.*

Proof. Suppose by contradiction that u has a self-intersection at a point $p \neq c_j$ for every j: $p = u(t_*) = u(t_{**})$, $t_* < t_{**}$. Let (a,b) the connected component of $[0,1] \setminus T_c(u)$ containing t_*. We know that $u|_{(a,b)}$ is a classical solution of (4.14), in particular it is of class $\mathcal{C}^2$.

First we notice that, by the energy integral, $|\dot{u}(t)| > 0$ for every t such that $u(t) \in B_R(0)$, hence both $\dot{u}(t_*)$ and $\dot{u}(t_{**})$ are different from 0. Let us define $v : [0,1] \to \mathbb{R}^2$ as follows

$$v(t) = \begin{cases} u(t) & t \in [0, t_*] \cup (t_{**}, 1], \\ u\left(\frac{t-t_*}{t_{**}-t_*} t_* + \left(1 - \frac{t-t_*}{t_{**}-t_*}\right) t_{**}\right) & t \in (t_*, t_{**}]. \end{cases}$$



The function v parametrizes a path with $u([0,1]) = v([0,1])$, but it goes along the loop connecting $u(t_*)$ and $u(t_{**})$ with the reversed orientation. The key observation is that this operation does not change the parity of the winding numbers with respect to the centres. Hence $v \in K_l$. Note that v is also an extremal for M, since $M(u) = M(v)$. On the other hand, it is trivially checked that, unless $\dot{u}(t_*) = \dot{u}(t_{**}) = 0$, v isn't $\mathcal{C}^1$ at those instants. So we have a new minimizer of M on K_l, which is collisions-free in an interval $(a,d) \ni t_*$, and hence here should be a classical solution of (4.14); but this isn't possible since $v|_{(a,d)} \notin \mathcal{C}^1((a,d), \mathbb{R}^2)$. $\square$

Coming back to the properties of $T_c(u)$, we state the following known result (see e.g. [4]).

Lemma 4.25. *If u has a collision at an instant $t_0 \in [0,1]$, then t_0 is isolated in $T_c(u)$. In particular, the cardinality of $T_c(u)$ is finite.*

Proof. Assume by contradiction that t_0 is an accumulation point in the set $T_c(u)$, with $u(t_0) = c_j$. By continuity, only collisions in c_j can accumulate in t_0. In this case there exists a sequence of intervals $((a_n, b_n))$ with $(a_n, b_n) \subset [0,1]$, $a_n \to t_0$ and $b_n \to t_0$ as $n \to \infty$, $u(a_n) = c_j = u(b_n)$ for every n, and

$$|u(t) - c_j| > 0 \quad \forall t \in (a_n, b_n).$$

On each of these intervals, since u is close to c_j (at least for n sufficiently large),

$$|u(t) - c_k| \geq C > 0 \qquad \text{for every } k \in \{1, \dots, N\},\ k \neq j.$$

Let us set $I(t) := |u(t) - c_j|^2$. Since $t \mapsto u(t)$ is a classical solution of (4.14) for $t \in (a_n, b_n)$, by differentiating twice $I(t)$ we obtain a modified Lagrange-Jacobi identity:

$$\ddot{I}(t) = -\frac{4}{\omega^2} + \frac{2}{\omega^2}(2-\alpha)\frac{m_j}{\alpha|u(t)-c_j|^\alpha} + \frac{2}{\omega^2}\sum_{\substack{k=1\\k\neq j}}^N \frac{m_k}{|u(t)-c_k|^\alpha}\left(\frac{2}{\alpha} - \frac{\langle u(t)-c_k, u(t)-c_j\rangle}{|u(t)-c_k|^2}\right).$$

Let $\xi_n \in (a_n, b_n)$ the maximizer of I in (a_n, b_n). It results $\ddot{I}(\xi_n) \leq 0$ for every n. Since in a neighbourhood of t_0 the second term in the expression of $\ddot{I}$ becomes arbitrarily large, while the other terms are bounded, we also get

$$\lim_{n\to\infty} \ddot{I}(\xi_n) = +\infty,$$

a contradiction. The collisions are isolated and, by compactness, the interval $[0,1]$ contains only a finite number of them. $\square$

Remark 4.26. The previous proof shows that, *if u collides in c_j, in a sufficiently small neighbourhood of c_j the function $I(t) = |u(t) - c_j|^2$ is strictly convex.*

Step 2). We would pass from a global analysis of the minimizer u to a local study in a neighbourhood of a collision. This is possible thanks to step 1: *u has an isolated collision at t_0 in a centre c_j, $j \in \{1,\dots,N\}$.* In particular there exist $c, d \in [0,1]$ such that

- $c < t_0 < d$ and t_0 is the unique collision time in $[c,d]$,
- the function I is strictly convex in $[c,d]$.

Let us set $\bar{p}_1 := u(c)$, $\bar{p}_2 = u(d)$. Since $u \in \mathcal{C}([c,d],\mathbb{R}^2)$ there exists $\mu > 0$ such that

$$|u(t) - c_k| \geq 2\mu > 0 \quad \text{for every } t \in [c,d] \text{ and for every } k \in \{1,\dots,N\} \setminus \{j\}.$$

This motivates us to write

$$V_\epsilon(y) = \frac{m_j}{\alpha|y-c_j|^\alpha} + V_\epsilon^j(y), \quad \text{where} \quad V_\epsilon^j(y) := \sum_{\substack{k=1\\k\neq j}}^{N} \frac{m_k}{\alpha|y-c_k|^\alpha}. \tag{4.15}$$

Indeed, in a neighbourhood U_j of c_j such that $\operatorname{dist}(U_j, c_k) \geq \mu$ for every k, the potential V_ϵ splits in a principal component due to the attraction of c_j, and a perturbation term V_ϵ^j due to the attraction of the other centres. Of course, for $x \in U_j$, V_ϵ^j is smooth and bounded.

We define

$$\begin{aligned}\widehat{\mathcal{K}}_l^{\bar{p}_1\bar{p}_2} := \Big\{ v \in H^1\left([c,d], \mathbb{R}^2 \setminus \{c_1,\dots,c_N\}\right) : v(c) = \bar{p}_1, v(d) = \bar{p}_2, \\ \text{the function } \Gamma_v(t) := \begin{cases} u(t) & t \in [0,c) \cup (d,1] \\ v(t) & t \in [c,d] \end{cases} \text{belongs to } K_l \Big\},\end{aligned}$$

and

$$\mathcal{K}_l^{\bar{p}_1\bar{p}_2} := \widehat{\mathcal{K}}_l^{\bar{p}_1\bar{p}_2} \cup \left\{ v \in H^1([c,d],\mathbb{R}^2) : v(c) = \bar{p}_1, v(d) = \bar{p}_2, \Gamma_v \in \mathfrak{Coll}_l \right\}.$$

The set $\mathcal{K}_l^{\bar{p}_1\bar{p}_2}$ is weakly closed. We define the restriction of the Maupertuis' functional to $\mathcal{K}_l^{\bar{p}_1\bar{p}_2}$ as

$$M_l^{\bar{p}_1\bar{p}_2} \,:\, \mathcal{K}_l^{\bar{p}_1\bar{p}_2} \to \mathbb{R} \cup \{+\infty\} \qquad M_l^{\bar{p}_1\bar{p}_2}(u) = \frac{1}{2}\int_c^d |\dot{v}(t)|^2\,dt \int_c^d (V_\epsilon(v(t)) - 1)\,dt.$$

It inherits the properties of weak lower semi-continuity and coercivity from M, then has a minimum on $\mathcal{K}_l^{\bar{p}_1\bar{p}_2}$ at a positive level. Since u is a minimizer of M on K_l, then $u|_{[c,d]}$ is a minimizer of $M_l^{\bar{p}_1\bar{p}_2}$ on $\mathcal{K}_l^{p_1p_2}$ (see Proposition 4.8).

Step 3). We introduce some more notation. For $\rho \geq 0$, we define

$$d(\rho) := \min\left\{ M_l^{\bar{p}_1\bar{p}_2}(v) : v \in \mathcal{K}_l^{\bar{p}_1\bar{p}_2}, \min_{t\in[c,d]} |v(t) - c_j| = \rho \right\}.$$

The value $d(0)$ is the minimum of $M_l^{\bar{p}_1\bar{p}_2}$ on the elements of $\mathcal{K}_l^{\bar{p}_1\bar{p}_2}$ which collide in c_j; hence $d(0)$ is achieved by $u|_{[c,d]}$.

Lemma 4.27. *The function $\rho \mapsto d(\rho)$ is continuous in $\rho = 0$.*

Proof. The proof is exactly the same of that of Lemma 17 in [26]. We have to take into account that in our case collisions occur in c_j and not in 0, and that we are dealing with the Maupertuis' functional and not with the action functional; nevertheless the same argument works. □

Now, given $0 < \rho_1 < \rho_2$, we set

$$\mathcal{K}_l^{\bar{p}_1\bar{p}_2}(\rho_1,\rho_2) := \left\{ v \in \mathcal{K}_l^{\bar{p}_1\bar{p}_2} : \min_{t\in[c,d]} |v(t) - c_j| \in [\rho_1,\rho_2] \right\}.$$

It is a weakly closed subset of $\mathcal{K}_l^{\bar{p}_1\bar{p}_2}$, so the restriction of $M_l^{\bar{p}_1\bar{p}_2}$ to $\mathcal{K}_l^{p_1p_2}(\rho_1,\rho_2)$ has a minimum that we denote as

$$m(\rho_1,\rho_2) := \min_{v\in\mathcal{K}_l^{\bar{p}_1\bar{p}_2}(\rho_1,\rho_2)} M_l^{\bar{p}_1\bar{p}_2}(v).$$

We also set

$$\mathcal{M}_{\rho_1\rho_2} := \left\{ v \in \mathcal{K}_l^{\bar{p}_1\bar{p}_2}(\rho_1,\rho_2) : M_l^{\bar{p}_1\bar{p}_2}(v) = m(\rho_1,\rho_2) \text{ and } \min_{t\in[c,d]} |v(t) - c_j| < \rho_2 \right\}.$$

In this step we aim at proving the following result.

Proposition 4.28. *There exists $\bar{\rho} > 0$ such that for $\rho_1, \rho_2 : 0 < \rho_1 < \rho_2 \leq \bar{\rho}$ implies $\mathcal{M}_{\rho_1 \rho_2} = \emptyset$.*

Remark 4.29. The proposition states that if we force the functions to go very close to c_j, i.e.

$$\min_{t \in [c,d]} |v(t) - c_j| < \bar{\rho},$$

then the minima $m(\rho_1, \rho_2)$ are achieved by elements of $\mathcal{K}_l^{\bar{p}_1 \bar{p}_2}(\rho_1, \rho_2)$ which stay as far as possible from c_j.

Assume by contradiction that the statement is not true. Then there existed two sequences (ρ_n), $(\bar{\rho}_n)$ such that

$$\begin{gathered} 0 < \rho_n < \bar{\rho}_n \quad \forall n, \qquad \rho_n \to 0, \bar{\rho}_n \to 0, \quad \text{for } n \to \infty, \\ \forall n \; \exists u_n \in \mathcal{K}_l^{\bar{p}_1 \bar{p}_2} : \min_{t \in [c,d]} |u_n(t) - c_j| = \rho_n, \\ M_l^{\bar{p}_1 \bar{p}_2}(u_n) = m(\rho_n, \bar{\rho}_n) = d(\rho_n). \end{gathered} \tag{4.16}$$

We can assume also that for every $n \in \mathbb{N}$

$$\max \left\{ \inf_{y \in \partial B_{\rho_n}(c_j)} |\bar{p}_1 - y|, \inf_{y \in \partial B_{\rho_n}(c_j)} |\bar{p}_2 - y| \right\} > 0.$$

Thanks to Lemma 4.27, $M_l^{\bar{p}_1 \bar{p}_2}(u_n) \to d(0)$ for $n \to \infty$, namely (u_n) is a minimizing sequence in $\mathcal{K}_l^{\bar{p}_1 \bar{p}_2}$ (we are assuming that the minimum of $M_l^{\bar{p}_1 \bar{p}_2}$ is achieved over collisions). Since $M_l^{\bar{p}_1 \bar{p}_2}$ is coercive, (u_n) is bounded and up to subsequence is weakly convergent to a function $\widetilde{u} \in \mathcal{K}_l^{\bar{p}_1 \bar{p}_2}$, which is a minimizer of $M_l^{\bar{p}_1 \bar{p}_2}$ (possibly different from $u|_{[c,d]}$) due to the weakly lower semi-continuity of $M_l^{\bar{p}_1 \bar{p}_2}$. We point out that $\widetilde{u}$ has to collide in c_j and could collide in centres different from c_j as well.
By Lemma 4.3, the energy of $\widetilde{u}$ is constant and equal to $-1/\widetilde{\omega}^2$, where

$$\widetilde{\omega}^2 := \frac{\int_d^c V_\epsilon(\widetilde{u}) - 1}{\frac{1}{2} \int_c^d |\dot{\widetilde{u}}|^2}.$$

Now, the same discussion of step 1 shows that the set $T_c(\widetilde{u})$ of collision times of $\widetilde{u}$ contains a finite number of elements, and we can assume that

- there exists a unique collision time t_0 in $[c,d]$ such that $\widetilde{u}(t_0) = c_j$,
- there exists $\mu > 0$ such that $|\widetilde{u}(t) - c_k| \geq 2\mu > 0$ for every $t \in [c,d]$, for every $k \neq j$.
- the function $|\widetilde{u}(t) - c_j|^2$ is strictly convex in $[c,d]$.

Otherwise we can replace $[c,d]$ with a smaller interval.

The paths u_n enjoy some common properties. Firstly, since the weak convergence in H^1 implies the uniform one, there exists $n_0 \in \mathbb{N}$ such that

$$n \geq n_0 \Rightarrow |u_n(t) - c_k| \geq \mu \quad \forall t \in [c,d], \; \forall k \neq j. \tag{4.17}$$

We rename as (u_n) the sequence obtained by dropping the first $(n_0 - 1)$-terms. Let us set

$$T_{\rho_n}(u_n) = \{ t \in [c,d] : |u_n(t) - c_j| = \rho_n \}.$$

We also introduce the polar coordinates and the (absolute value of the) angular momentum of u_n with respect to the centre c_j:

$$\begin{gathered} u_n(t) = c_j + w_n(t) e^{i \phi_n(t)}, \\ \mathfrak{C}_n^j(t) := |(u_n(t) - c_j) \wedge \dot{u}_n(t)|. \end{gathered}$$

Here $w_n : [c,d] \to \mathbb{R}^+$ and $\phi_n(t) : [c,d] \to \mathbb{R}$.

Lemma 4.30. *For every $n \in \mathbb{N}$ the path u_n has the following properties:*

(i) *If (c', d') is a connected component of $[c,d] \setminus T_{\rho_n}(u_n)$, then $u_n|_{(c',d')}$ is $\mathcal{C}^2$ and solves*

$$\omega_n^2 \ddot{u}_n(t) = \nabla V_\epsilon(u_n(t)) \quad \textit{where} \quad \omega_n^2 := \frac{\int_c^d (V_\epsilon(u_n) - 1)}{\frac{1}{2} \int_c^d |\dot{u}_n|^2}. \tag{4.18}$$

(*ii*) *For every* $n \in \mathbb{N}$, *there exist* $t_n^- \leq t_n^+$ *such that:*

$$\begin{aligned} |u_n(t) - c_j| &> \rho_n \qquad t \in [c, t_n^-) \cup (t_n^+, d] \\ |u_n(t) - c_j| &= \rho_n \qquad t \in [t_n^-, t_n^+], \end{aligned}$$

namely $T_{\rho_n}(u_n) = [t_n^-, t_n^+]$.

(*iii*) *The sequence* (ω_n^2) *is bounded above and uniformly bounded below by a strictly positive constant. Hence there exist a subsequence of* (u_n) *(still denoted* (u_n)*) and* $\Omega > 0$ *such that*

$$\lim_{n \to \infty} \omega_n = \Omega.$$

(*iv*) *The energy of the function* u_n *is constant in* $[c, d]$*:*

$$\frac{1}{2}|\dot{u}_n(t)|^2 - \frac{V_\epsilon(u_n(t))}{\omega_n^2} = -\frac{1}{\omega_n^2} \qquad \forall t \in [c, d].$$

Moreover, the sequence $(-1/\omega_n^2)$ *is bounded in* $\mathbb{R}$.

(*v*) *The function* $\phi_n|_{(t_n^-, t_n^+)}$ *is* $\mathcal{C}^2$, *strictly monotone and is a solution of*

$$\ddot{\phi}_n(t) = \frac{1}{\rho_n \omega_n^2} \left\langle \nabla V_\epsilon \left(c_j + \rho_n e^{i\phi_n(t)}\right), i e^{i\phi_n(t)} \right\rangle. \tag{4.19}$$

(*vi*) *A minimizer of* $M_l^{\bar{p}_1 \bar{p}_2}$ *in* $\mathcal{K}_l^{\bar{p}_1 \bar{p}_2}$ *is of class* $\mathcal{C}^1$ *in* $[c, d]$. *In particular, this holds true for* u_n, *for every* n.

Proof. The proof of (i) and (v) is the same of the points (i) and (ii) of Lemma 4.19, respectively.
(ii) On every interval $(c', d') \subset (c, d) \setminus T_{\rho_n}(u_n)$ u_n solves the (4.18); hence the uniform convergence of (u_n) to $\widetilde{u}$ and the computation of

$$\frac{d^2}{dt^2}|u_n(t) - c_j|^2$$

(see the proof of Lemma 4.25) imply that the function $|u_n(t) - c_j|^2$ is strictly convex over such an interval. Therefore, if there exist $t_1 < t_2$ such that $|u_n(t_1) - c_j| = |u_n(t_2) - c_j| = \rho_n$ then $|u_n(t) - c_j| = \rho_n$ for every $t \in (t_1, t_2)$.
(iii) We have

$$\omega_n^2 = \frac{M_l^{\bar{p}_1 \bar{p}_2}(u_n)}{\frac{1}{4}\left(\int_c^d |\dot{u}_n|^2\right)^2} = \frac{d(\rho_n)}{\frac{1}{4}\|\dot{u}_n\|^4_{L^2([c,d])}}. \tag{4.20}$$

We know that

$$0 < d(0) < d(\rho_n) \quad \text{and} \quad d(\rho_n) \to d(0) \Rightarrow \exists C_1, C_2 > 0 : \; C_1 \leq d(\rho_n) \leq C_2 \; \forall n. \tag{4.21}$$

As to the denominator of (4.20), observe that, for every, n the path u_n covers at least a fixed distance; therefore, like in the proof of Lemma 4.16, there exists $C_3 > 0$ such that

$$\|\dot{u}_n\|_{L^2([c,d])} \geq C_3 \qquad \forall n \tag{4.22}$$

Moreover, being (u_n) a minimizing sequence of a coercive functional, (u_n) is bounded in the H^1-norm and a fortiori there exists $C_4 > 0$ such that

$$\|\dot{u}_n\|_{L^2([c,d])} \leq C_4 \qquad \forall n. \tag{4.23}$$

Altogether, (4.20), (4.21), (4.22) and (4.23) imply the assertion.
(iv) The energy is constant, as proved in Lemma 4.3. The boundedness of $(-1/\omega_n^2)$ is a trivial consequence of point (iii).
(vi) We take advantage again of the Proposition 3.16 in [2]: since u_n is a minimizer of $M_l^{\bar{p}_1 \bar{p}_2}$ on $K_l^{\bar{p}_1 \bar{p}_2}(\rho_n, \bar{\rho}_n)$, then one of the following situations occurs:

a) $t_n^- < t_n^+$ and $u \in \mathcal{C}^1([c, d])$,
b) $t_n^- = t_n^+$ and $u \in \mathcal{C}^1([c, d])$,
c) $t_n^- = t_n^+$ and $u \notin \mathcal{C}^1([c, d])$.

But applying the line of reasoning already used in the proof of Theorem 4.22, it is easy to check that the situation c) cannot occur. □

We are now in a position to prove the following result.

Proposition 4.31. *The minimizer u_n is free of self-intersections in $[c,d]$. In particular, the total variation of the angle ϕ_n is smaller then 2π.*

Proof. The function u_n has no self-intersections for $t \in [c, t_n^-) \cup (t_n^+, d]$. The prove is the same of that of Proposition 4.24. If u_n has a self-intersection on the obstacle $\{|y - c_j| = \rho_n\}$, the monotonicity of ϕ_n implies that u_n makes a complete wind around it. But then we can consider the function v which parametrizes the same path of u, but reverses the orientation on the obstacle. One has $M_l^{\bar{p}_1\bar{p}_2}(u_n) = M_l^{\bar{p}_1\bar{p}_2}(v)$, so that v is a local minimizer of $M_l^{\bar{p}_1\bar{p}_2}$ with $\min_{t\in[c,d]} |v(t) - c_j| = \rho_n$. Hence v satisfies the energy integral, so that it cannot approach to the obstacle with velocity 0. Therefore it should be a minimizer which is not $\mathcal{C}^1$, against point (vi) of the previous lemma. □

Proposition 4.32. *The estimates*

$$\mathfrak{C}_n^j(t) = \rho_n^{\frac{2-\alpha}{2}} \sqrt{\frac{2m_j}{\omega_n^2 \alpha}} \left(1 + O(\rho_n^\alpha)\right), \qquad t_n^+ - t_n^- = O(\rho_n^{\frac{\alpha+2}{2}})$$

hold for $n \to \infty$.

Proof. Since $u_n \in \mathcal{C}^1([c,d])$, it can lean against the obstacle $\{|y - c_j| = \rho_n\}$ with velocity $\dot{u}_n(t)$ orthogonal to the radial segment joining c_j and $u_n(t)$. Therefore for every $t \in [t_n^-, t_n^+]$ there holds $\mathfrak{C}_n^j(t) = \rho_n |\dot{u}_n(t)| = \rho_n^2 \dot{\phi}_n(t)$. From the expression of the energy and the uniform boundedness of $(V_\epsilon^j(u_n))$ (see equation (4.17)) we get

$$\begin{aligned} \mathfrak{C}_n^j(t) &= \rho_n \sqrt{\frac{2}{\omega_n^2}\left(\frac{m_j}{\alpha\rho_n^\alpha} + V_\epsilon^j\left(u_n(t)\right) - 1\right)} \\ &= \rho_n^{\frac{2-\alpha}{2}} \sqrt{\frac{2m_j}{\omega_n^2\alpha} + \frac{2\rho_n^\alpha}{\omega_n^2}\left(V_\epsilon^j(u_n(t)) - 1\right)} = \rho_n^{\frac{2-\alpha}{2}} \sqrt{\frac{2m_j}{\omega_n^2\alpha}} \left(1 + O(\rho_n^\alpha)\right). \end{aligned} \tag{4.24}$$

Therefore

$$\dot{\phi}_n(t) = \rho_n^{\frac{-2-\alpha}{2}} \sqrt{\frac{2m_j}{\omega_n^2\alpha}} \left(1 + O(\rho_n^\alpha)\right),$$

and the total variation of ϕ_n on the obstacle is

$$\phi_n(t_n^+) - \phi_n(t_n^-) = \rho_n^{\frac{-2-\alpha}{2}} \sqrt{\frac{2m_j}{\omega_n^2\alpha}} \left(1 + O(\rho_n^\alpha)\right)(t_n^+ - t_n^-).$$

This variation is bounded by 2π, so that $t_n^+ - t_n^- = O(\rho_n^{\frac{\alpha+2}{2}})$. □

In order to obtain a contradiction, we consider a blow-up of our sequence.
For every $n \in \mathbb{N}$, let us fix $t_n \in [t_n^-, t_n^+]$. By the previous Proposition the sequence (t_n) tends to the limit t_0 which is the unique collision time of $\widetilde{u}$ in (c,d). Let us set

$$c_n := \rho_n^{-\frac{\alpha+2}{2}}(c - t_n), \qquad d_n := \rho_n^{-\frac{\alpha+2}{2}}(d - t_n).$$

We also define

$$s_n^- := \rho_n^{-\frac{\alpha+2}{2}}(t_n^- - t_n), \qquad s_n^+ := \rho_n^{-\frac{\alpha+2}{2}}(t_n^+ - t_n)$$

We note that $c_n \to -\infty$, $d_n \to +\infty$ as $n \to \infty$. As far as (s_n^-) and (s_n^+) are concerned, they are two bounded sequences thanks to proposition 4.32, so that there exists a subsequence of (ρ_n) (which we still denote (ρ_n)) such that they converge to limits s^- and s^+ respectively.

Remark 4.33. Consider the change of variable

$$s(t,n) = \rho_n^{-\frac{\alpha+2}{2}}(t - t_n) \Leftrightarrow t(s,n) = t_n + \rho_n^{\frac{\alpha+2}{2}} s.$$

One has

$$s(t,n) \in [c_n, d_n] \Leftrightarrow t(s,n) \in [c,d], \qquad s(t,n) \in [s_n^-, s_n^+] \Leftrightarrow t(s,n) \in [t_n^-, t_n^+].$$

We introduce the sequence of paths $v_n : [c_n, d_n] \to \mathbb{R}^2$,

$$v_n(s) := c_j + \frac{1}{\rho_n}\left(u_n\left(t_n + \rho_n^{\frac{\alpha+2}{2}} s\right) - c_j\right).$$

In polar coordinates with respect to the centre c_j we write

$$v_n(s) = c_j + \bar{w}_n(s) e^{i\bar{\phi}_n(s)},$$

where

$$\bar{w}_n(s) = \frac{1}{\rho_n} w_n\left(t_n + \rho_n^{\frac{\alpha+2}{2}} s\right), \qquad \bar{\phi}_n(s) = \phi_n\left(t_n + \rho_n^{\frac{\alpha+2}{2}} s\right).$$

Each v_n is of class $\mathcal{C}^1$ and

$$\begin{aligned} |v_n(s) - c_j| &= 1 && \text{for } s \in [s_n^-, s_n^+], \\ |v_n(s) - c_j| &> 1 && \text{for } s \in [c_n, s_n^-) \cup (s_n^+, d_n]. \end{aligned}$$

The restriction $v_n|_{[c_n, s_n^-)\cup(s_n^+, d_n]}$ is of class $\mathcal{C}^2$ and satisfies the equation

$$\begin{aligned} \ddot{v}_n(s) &= -\frac{\rho_n^{2+\alpha}}{\omega_n^2 \rho_n} \sum_{k=1}^N \frac{m_k}{|u_n(t(s,n)) - c_k|^{\alpha+2}} \left(u_n(t(s,n)) - c_k\right) \\ &= -\frac{m_j \left[\frac{1}{\rho_n}\left(u_n(t(s,n)) - c_j\right) \pm c_j\right]}{\omega_n^2 \left|\frac{1}{\rho_n}\left(u_n(t(s,n)) - c_j\right) \pm c_j\right|^{\alpha+2}} + \frac{\rho_n^{\alpha+1}}{\omega_n^2} \nabla V_\epsilon^j(u_n(t(s,n))) \\ &= -\frac{m_j}{\omega_n^2 |v_n(s) - c_j|^{\alpha+2}} \left(v_n(s) - c_j\right) + O(\rho_n^{\alpha+1}). \end{aligned}$$

This suggests to consider the quantity

$$\bar{h}_n(s) := \frac{1}{2}|\dot{v}_n(s)|^2 - \frac{m_j}{\omega_n^2 \alpha |v_n(s) - c_j|^\alpha},$$

the energy of the function v_n for the potential of the α-Kepler's problem with centre in c_j. This is not a constant function in $[c_n, d_n]$, however it can be easily controlled.

$$\begin{aligned} \bar{h}_n(s) &= \rho_n^\alpha \left[\frac{1}{2}\left|\dot{u}_n(t(s,n))\right|^2 - \frac{m_j}{\omega_n^2 \alpha |u_n(t(s,n)) - c_j|^\alpha}\right] \\ &= \rho_n^\alpha \left[-\frac{1}{\omega_n^2} + \frac{1}{\omega_n^2} V_\epsilon^j\left(u_n(t(s,n))\right)\right]. \end{aligned}$$

Therefore, form the point (iv) of Lemma 4.30 we deduce

$$\lim_{n\to\infty} \bar{h}_n(s) = 0 \quad \text{for every } s \in [c_n, d_n].$$

The uniform boundedness of $\left(V_\epsilon^j(u_n)\right)$ makes the convergence uniform on every closed interval $[a, b] \subset \mathbb{R}$. Let us also define the (absolute value of the) angular momentum of v_n with respect to the centre c_j:

$$\bar{\mathfrak{C}}_n^j(s) := \left|(v_n(s) - c_j) \wedge \dot{v}_n(s)\right|.$$

If $s \in [s_n^-, s_n^+]$, using Proposition 4.32 we obtain

$$\bar{\mathfrak{C}}_n^j(s) = \rho_n^{\frac{\alpha+2}{2}} \dot{\phi}_n(t(s,n)) = \rho_n^{\frac{\alpha-2}{2}} \mathfrak{C}_n^j(t(s,n)) = \sqrt{\frac{2m_j}{\omega_n^2 \alpha}} \left(1 + O(\rho_n^\alpha)\right).$$

Hence

$$\lim_{n\to\infty} \bar{\mathfrak{C}}_n^j(s) = \sqrt{\frac{2m_j}{\Omega^2 \alpha}}, \quad \text{for every } s \in [s_-, s_+], \tag{4.25}$$

with uniform convergence in $[s^-, s^+]$. For the reader's convenience, we recall that $\Omega = \lim_n \omega_n$. The previous computation implies that the sequence $(\bar{\mathfrak{C}}_n^j|_{[s^-,s^+]})$ is uniformly bounded in a neighbourhood of $[s^-, s^+]$.

Recalling the point (v) of Lemma 4.30, we obtain an equation for $\bar{\phi}_n$ when $s \in (s_n^-, s_n^+)$:

$$\begin{aligned}\ddot{\bar{\phi}}_n(s) &= \frac{\rho_n^{\alpha+1}}{\omega_n^2}\left\langle \nabla V_\epsilon\left(c_j + \rho_n e^{i\bar{\phi}_n(s)}\right), ie^{i\bar{\phi}_n(s)}\right\rangle \\ &= -\frac{1}{\omega_n^2}\left\langle m_j e^{i\bar{\phi}_n(s)}, ie^{i\bar{\phi}_n(s)}\right\rangle + \frac{\rho_n^{\alpha+1}}{\omega_n^2}\left\langle \nabla V_\epsilon^j\left(c_j + \rho_n e^{i\bar{\phi}_n(s)}\right), ie^{i\bar{\phi}_n(s)}\right\rangle \\ &= 0 + O(\rho_n^{\alpha+1}).\end{aligned}$$

Hence the restriction $v_n|_{(s_n^-, s_n^+)}$ is of class $\mathcal{C}^2$ and satisfies

$$\begin{aligned}\ddot{v}_n(s) = \ddot{\bar{\phi}}_n(s) ie^{i\bar{\phi}_n(s)} - \left(\dot{\bar{\phi}}_n(s)\right)^2 e^{i\bar{\phi}_n(s)} = \ddot{\bar{\phi}}_n(s) i\left(v_n(s) - c_i\right) - \left(\bar{\mathfrak{C}}_n^j(s)\right)^2\left(v_n(s) - c_i\right) = \\ = -\left(\bar{\mathfrak{C}}_n^j(s)\right)^2\left(v_n(s) - c_i\right) + i\left(v_n(s) - c_i\right) O(\rho_n^{\alpha+1}).\end{aligned}$$

Summing up

$$\ddot{v}_n = \begin{cases} -\frac{m_j(v_n - c_i)}{\omega_n^2 |v_n - c_i|^{\alpha+2}} + O(\rho_n^{\alpha+1}) & \text{in } [c_n, s_n^-) \cup (s_n^+, d_n] \\ \\ -\left(\bar{\mathfrak{C}}_n^j\right)^2 (v_n - c_i) + i\left(v_n - c_i\right) O(\rho_n^{\alpha+1}) & \text{in } (s_n^-, s_n^+).\end{cases} \tag{4.26}$$

This shows that v_n is not necessarily of class $\mathcal{C}^2$ in s_n^- and s_n^+; anyway there exist the right and left limits at these points.

Proposition 4.34. *Let $[a,b] \subset \mathbb{R}$, $a \le 0 \le b$. There exists a subsequence of (v_n) which converges in the $\mathcal{C}^1$ topology on $[a,b]$.*

Proof. There is uniform convergence to 0 of the energies $\bar{h}_n$ over $[a,b]$; thus the restrictions $(\bar{h}_n|_{[a,b]})$ define a bounded sequence in the uniform topology. Since for every n

$$\inf_{s \in [a,b]} |v_n(s) - c_j| = |v_n(0) - c_j| = 1,$$

for every $s \in [a,b]$

$$|\dot{v}_n(s)|^2 = 2\bar{h}_n(s) + \frac{2m_j}{\omega_n^2 \alpha\, |v_n(s) - c_j|^\alpha} \le 2\|\bar{h}_n|_{[a,b]}\|_\infty + 2\frac{m_j}{\omega_n^2 \alpha}.$$

Therefore

$$\|\dot{v}_n|_{[a,b]}\|_\infty \le \sqrt{2}\sup_n \left(\|\bar{h}_n|_{[a,b]}\|_\infty + \frac{m_j}{\omega_n^2 \alpha}\right)^{\frac{1}{2}} < +\infty,$$

i.e. $\left(\dot{v}_n|_{[a,b]}\right)$ is uniformly bounded. Now,

(1) $(v_n|_{[a,b]})$ is equicontinuous: for every $s_1, s_2 \in [a,b]$, for every $n \in \mathbb{N}$

$$|v_n(s_1) - v_n(s_2)| \le \|\dot{v}_n|_{[a,b]}\|_\infty |s_1 - s_2| \le C|s_1 - s_2|.$$

(2) $\left(v_n|_{[a,b]}\right)$ is uniformly bounded: for every $s \in [a,b]$, for every $n \in \mathbb{N}$:

$$|v_n(s)| \le |v_n(0)| + C|s| \le \epsilon + 1 + C\max\{|a|, |b|\}.$$

Hence we can apply the Ascoli-Arzelà theorem, to obtain a uniformly converging subsequence (still denoted by (v_n)). From equation (4.26) we see also that $(\ddot{v}_n|_{[a,b]})$ is uniformly bounded. Indeed

$$\begin{aligned}&|\ddot{v}_n(s)| \le \frac{m_j}{\omega_n^2} + O(\rho_n^{\alpha+1}) \le C < +\infty \qquad \text{for every } s \in [c_n, s_n^-) \cup (s_n^+, d_n], \\ &|\ddot{v}_n(s)| \le \left(\bar{\mathfrak{C}}_n^j(s)\right)^2 + O(\rho_n^{\alpha+1}) \le C < +\infty \qquad \text{for every } s \in (s_n^-, s_n^+) \\ &\max\left\{\lim_{s \to (s_n^\pm)^\pm} |\ddot{v}(s)|\right\} = C < +\infty,\end{aligned}$$

(recall (4.25) for the second bound) and immediately $\sup_n \|\ddot{v}_n|_{[a,b]}\|_\infty < +\infty$. Moreover

$$\lim_{n\to\infty} \frac{1}{2}|\dot{v}_n(0)|^2 = \lim_{n\to\infty} \bar{h}_n(0) + \frac{m_j}{\omega_n^2 \alpha} = \frac{m_j}{\Omega^2 \alpha}.$$

In particular, $(\dot v_n(0))$ is bounded, too. Now it is sufficient to repeat the previous argument and use the Ascoli-Arzelà theorem for $(\dot v_n)$. $\square$

Applying the Proposition on each interval $[-k,k]$ we obtain a subsequence of (v_n) (still denoted by (v_n)) which converges in the $\mathcal{C}^1$ topology on every closed interval of $\mathbb{R}$. We call $v:\mathbb{R}\to\mathbb{R}^2$ its limit. By (4.26) the sequence $(\ddot v_n)$ uniformly converges on every compact subset of $\mathbb{R}\setminus\{s^-,s^+\}$, so $v\in\mathcal{C}^2\left(\mathbb{R}\setminus\{s^-,s^+\}\right)$ and

- v is a classical solution of the α-Kepler's problem
$$\ddot v(s) = -\frac{m_j}{\Omega^2|v(s)-c_j|^{\alpha+2}}\left(v(s)-c_j\right) \quad \text{for } s\in(-\infty,s^-)\cup(s^+,+\infty).$$
- v has constant energy equal to 0 (even in $[s^-,s^+]$),
- v has constant angular momentum with respect to c_j, equal to $\bar{\mathfrak{C}}^j=\sqrt{\frac{2m_j}{\Omega^2\alpha}}$ (even in $[s^-,s^+]$),
- $|v(s)-c_j|=1$ for $s\in[s^-,s^+]$,
- $|v(s)-c_j|>1$ for $s\in(-\infty,s^-)\cup(s^+,+\infty)$.

We write $v(s)=c_j+w(s)\exp\{i\phi(s)\}$, and term $\phi^-:=\phi(s^-)$, $\phi^+:=\phi(s^+)$. Thanks to the conservation of the angular momentum, the function $s\mapsto\phi(s)$ is strictly monotone; it is not restrictive to assume that it is increasing, and it makes sense to write

$$\phi(+\infty)=\lim_{s\to+\infty}\phi(s), \qquad \phi(-\infty)=\lim_{s\to-\infty}\phi(s).$$

Writing the energy in polar coordinates we get

$$ds=\frac{dw}{\sqrt{2\left(\frac{m_j}{\alpha\Omega^2w^\alpha}-\frac{\left(\bar{\mathfrak{C}}^j\right)^2}{w^2}\right)}}.$$

Hence

$$\phi(+\infty)-\phi^+=\int_{s^+}^{+\infty}\frac{d\phi}{ds}\,ds=\int_1^{+\infty}\frac{\bar{\mathfrak{C}}^j\,dw}{w^2\sqrt{\frac{2m_j}{\alpha\Omega^2w^\alpha}-\frac{\left(\bar{\mathfrak{C}}^j\right)^2}{w^2}}}=\int_1^{+\infty}\frac{dw}{w^2\sqrt{\frac{1}{w^\alpha}-\frac{1}{w^2}}}=\int_0^1\frac{d\xi}{\sqrt{\xi^\alpha-\xi^2}}.$$

The same computation holds true for $\phi^- - \phi(-\infty)$. With the change of variable $\xi=\eta^{\frac{2}{2-\alpha}}$ we obtain

$$\phi(+\infty)-\phi^+=\phi^- - \phi(-\infty)=\frac{2}{2-\alpha}\int_0^1\frac{\eta^{\frac{\alpha}{2-\alpha}}}{\sqrt{\eta^{\frac{2\alpha}{2-\alpha}}-\eta^{\frac{4}{2-\alpha}}}}\,d\eta=\frac{2}{2-\alpha}\int_0^1\frac{d\eta}{\sqrt{1-\eta^2}}=\frac{\pi}{2-\alpha}.$$

We deduce the following estimate for the total variation of the angle ϕ:

$$\phi(+\infty)-\phi(-\infty)=\frac{2\pi}{2-\alpha}+\phi^+-\phi^-\geq\frac{2\pi}{2-\alpha}. \tag{4.27}$$

On the other hand we know that $\bar\phi_n$ uniformly converges to ϕ on every closed interval $[a,b]$ of $\mathbb{R}$. For n sufficiently large

$$\bar\phi_n(b)-\bar\phi_n(a)\leq\bar\phi_n(d_n)-\bar\phi_n(c_n)<2\pi$$

for Proposition 4.31. Passing to the limit for $n\to\infty$

$$\phi(b)-\phi(a)\leq 2\pi.$$

Since a and b are arbitrarily chosen, we can take $a\to-\infty$, $b\to+\infty$ to obtain

$$\phi(+\infty)-\phi(-\infty)\leq 2\pi. \tag{4.28}$$

If $\alpha\in(1,2)$, (4.27) and (4.28) give a contradiction, and the proof of Proposition 4.28 is complete. When $\alpha=1$ we don't reach yet a contradiction, but each result of this step (except Proposition 4.28, of course) still holds true.

Step 4) Conclusion of the proof of Theorem 4.12 for $\alpha \in (1,2)$. From Proposition 4.28 there exists $\bar\rho > 0$ such that, if $0 < \rho_1 < \rho_2 < \rho^* \leq \bar\rho$,

$$u \text{ is a minimizer of } M_l^{\bar p_1 \bar p_2}|_{\mathcal{K}_l(\rho_2,\rho^*)} \Rightarrow \min_{t\in[c,d]} |v(t)-c_i| = \rho^*,$$

and

$$u \text{ is a minimizer of } M_l^{\bar p_1 \bar p_2}|_{\mathcal{K}_l(\rho_1,\rho_2)} \Rightarrow \min_{t\in[c,d]} |v(t)-c_i| = \rho_2.$$

Hence $d(\rho^*) < d(\rho_2) < d(\rho_1)$. We recall that the function $d(\cdot)$ is continuous in 0, so that taking $\rho_1 \to 0^+$ we obtain $d(\rho^*) < d(0)$: this is a contradiction, since we are assuming that the minimum of $M_l^{\bar p_1 \bar p_2}$ on $\mathcal{K}_l^{p_1 p_2}$ is achieved over collision paths. Applying the same argument for each collision time of u we obtain, by Proposition 4.8, that a minimizer of M on K_l is collisions-free, too.
Step 5) The case $\alpha = 1$. In case $\alpha = 1$ the third step does not give a contradiction: indeed it is possible that $\phi(+\infty) - \phi(-\infty) = 2\pi$. However, we will strongly use the results proved in step 3).

In order to complete the proof of Theorem 4.12, we need the following statement.

Proposition 4.35. *If the local minimizer $u \in K_l$ of M has a collision, then there exists a possibly different minimizer $\widehat u \in K_l$ such that the collision set $T_c(\widehat u)$ consists of a unique instant, and $y(t) := \widehat u(\omega t)$ is an ejection-collision solution of* (4.1). *In particular, this implies $p_1 = p_2$.*

We keep the same notations already introduced. Since u is the minimizer of M in K_l, the set $T_c(u)$ is finite and we set $t_0 = \min T_c(u)$. We can define $c, d \in [0,1]$, $\bar p_1, \bar p_2 \in \mathbb{R}^2$, ... as in the previous steps. If there exists $\bar\rho > 0$ such that for every $0 < \rho_1 < \rho_2 < \bar\rho$ it result $\mathcal{M}_{\rho_1\rho_2} = \emptyset$, then u is collisions-free in $[c,d]$. Otherwise there exist two sequences (ρ_n) and $(\bar\rho_n)$ converging to 0 such that

$$0 < \rho_n < \bar\rho_n \quad \forall n, \qquad \forall n\ \exists u_n \in \mathcal{K}_l^{\bar p_1 \bar p_2}:\ \min_{t\in[c,d]} |u_n(t) - c_j| = \rho_n \text{ and } M_l^{\bar p_1 \bar p_2}(u_n) = m(\rho_n, \bar\rho_n) = d(\rho_n). \tag{4.29}$$

We know that (u_n) uniformly converges to a collision path $\widetilde u \in \mathcal{K}_l^{\bar p_1 \bar p_2}$ (possibly different from u), which minimizes $M_l^{\bar p_1 \bar p_2}$. We define

$$\widehat u(t) := \begin{cases} u(t) & t \in [0,1] \setminus [c,d] \\ \widetilde u(t) & t \in [c,d] \end{cases}$$

Now we use a classical method to deal with singularities and to extend solutions beyond collisions, firstly introduced in 1920 by Levi-Civita in [21]. In performing the Levi–Civita regularization we see an $\mathbb{R}^2$-valued function as a function in $\mathbb{C}$ and we exploit the conformal equivariance of the problem. In fact, we lift the geodesics with respect to the Jacobi metric into geodesics on the Riemann surface.

Definition 4.2. (Local Levi-Civita transform). For every complex-valued continuous function u we define the set $\Lambda(u)$ of the continuous function q such that

$$u(t) = q^2(\tau(t)) + c_j,$$

where we reparametrize the time as

$$dt = |q(\tau)|^2\, d\tau.$$

We will denote with "$'$" the differentiation with respect to τ, and ∇_q the gradient in the Levi-Civita space. We remark that if a path u does not collide in c_j, then $\Lambda(u)$ consist in two elements $\pm\sqrt{u(\tau(t))) - c_j}$.

Actually, we perform the Levi-Civita-type transform along the sequence given in (4.29). It is convenient to define

$$S_n := \int_c^d \frac{dt}{|u_n(t) - c_j|}.$$

Lemma 4.36. *The sequence (S_n) is bounded above and bounded below by a strictly positive constant. Hence there exist a subsequence (still denoted (S_n)) and $\widetilde S > 0$ such that*

$$\lim_{n\to\infty} S_n = \widetilde S.$$

Proof. Assume by contradiction that (S_n) were not bounded above:

$$\limsup_{n\to\infty} \int_c^d \frac{dt}{|u_n(t) - c_j|} = +\infty.$$

In the proof of point (iii) of Lemma 4.30 we showed that

$$\liminf_{n\to\infty} \int_c^d |\dot{u}_n(t)|^2 \, dt > 0,$$

and hence $(M_l^{\bar{p}_1\bar{p}_2}(u_n))$ is unbounded, too, in contradiction with the fact that (u_n) is a minimizing sequence of a coercive functional. Furthermore, since

$$\int_c^d \frac{dt}{|u_n(t) - c_j|} \geq \frac{d-c}{R+\epsilon} > 0,$$

(S_n) is also bounded below by a positive constant. □

For every n, we define the set $\Lambda(u_n)$ of the continuous function q_n such that

$$u_n(t) = q_n^2(\tau(t)) + c_j \qquad dt = S_n |q_n(\tau)|^2 \, d\tau.$$

We also set

$$\widetilde{u}(t) = \widetilde{q}^{\,2}(\tau(t)) + c_j \qquad dt = \widetilde{S} |\widetilde{q}(\tau)|^2 \, d\tau.$$

We point out that the new time τ depends on n (we keep in mind this dependence, but we don't write it down to ease the notation). However, setting $\tau(c) = 0$ for every n, the right end of the interval of definition of each function q_n is

$$\int_0^{\tau(d)} d\tau = \frac{1}{S_n} \int_c^d \frac{dt}{|u_n(t) - c_j|} = 1,$$

so that q_n is defined over $[0,1]$ for every n. We set $\tau_n^- := \tau(t_n^-)$ and $\tau_n^+ := \tau(t_n^+)$ (recall that $t_n^- = \inf\{t \in [c,d] : |u_n(t) - c_j| = \rho_n\}$, $t_n^+ = \sup\{t \in [c,d] : |u_n(t) - c_j| = \rho_n\}$) .
Since u_n doesn't collide in c_j for every n, we can make a choice of $q_n \in \Lambda(u_n)$, in such a way that the sequence (q_n) is uniformly convergent to a path $\widetilde{q} \in \Lambda(\widetilde{u})$ (for instance $q_n = +\sqrt{u_n - c_j}$ for every n). The constraint $B_{\rho_n}(c_j)$ corresponds trough the transformation to the ball $B_{\sqrt{\rho_n}}(0)$, so that q_n satisfies

$$\begin{aligned} |q_n(\tau)| &> \sqrt{\rho_n} \qquad && \tau \in [0, \tau_n^-) \cup (\tau_n^+, 1] \\ |q_n(\tau)| &= \sqrt{\rho_n} \qquad && \tau \in [\tau_n^-, \tau_n^+]. \end{aligned}$$

In polar coordinates we write

$$q_n(\tau) = \kappa_n(\tau) e^{i\sigma_n(\tau)},$$

where $\kappa_n : [0,1] \to \mathbb{R}^+$, $\sigma_n : [0,1] \to \mathbb{R}$.
The next lemma establish a relationship between the variational properties of a function and its Levi-Civita transform.

Lemma 4.37. *Every $q_n \in \Lambda(u_n)$ is a local minimizer of*

$$\widetilde{M}(q) := 4 \int_0^1 |q'|^2 \int_0^1 \left[m_j + \left(V_\epsilon^j(q^2 + c_j) - 1 \right) |q|^2 \right]$$

at a strictly positive level.

Proof. It is sufficient to write the factors of M in terms of τ and q_n:

$$|\dot{u}_n(t)|^2 \, dt = \left| 2 q_n(\tau(t)) q_n'(\tau(t)) \frac{d\tau}{dt}(t) \right|^2 \, dt = \frac{4}{S_n} |q_n'(\tau)|^2 \, d\tau,$$

and

$$\begin{aligned} (V_\epsilon(u_n(t)) - 1) \, dt &= \left(\frac{m_j}{|q_n(\tau(t))|^2} + V_\epsilon^j(q_n^2(\tau(t)) + c_j) - 1 \right) dt \\ &= S_n \left[m_j + \left(V_\epsilon^j(q_n^2(\tau) + c_j) - 1 \right) |q_n(\tau)|^2 \right] \, d\tau. \end{aligned}$$

□

Remark 4.38. We get a functional of Maupertuis-type. In this case the potential is no more singular, and the mass m_j plays the role of the energy.

Now a technical result:

Lemma 4.39. *For every n, let*

$$\widetilde{\omega}_n^2 := \frac{\int_0^1 \left[m_j + \left(V_\epsilon^j(q_n^2+c_j)-1\right)|q_n|^2\right]}{\frac{1}{2}\int_0^1 |q_n'|^2}.$$

The sequence $(\widetilde{\omega}_n^2)$ is bounded above and bounded below by a strictly positive constant. Hence there exist a subsequence (still denoted $(\widetilde{\omega}_n)$) and $\widetilde{\Omega} > 0$ such that

$$\lim_{n\to\infty} \widetilde{\omega}_n = \widetilde{\Omega}.$$

Proof. There holds

$$\widetilde{\omega}_n^2 = \frac{\frac{1}{S_n}\int_c^d V_\epsilon(u_n) - 1}{\frac{S_n}{8}\int_c^d |\dot{u}_n|^2} = \frac{4}{S_n^2}\omega_n^2.$$

Now it is sufficient recall Lemma 4.36 and the fact that $\omega_n^2 \to \Omega^2 > 0$. □

From now on, we will always consider the subsequence introduced in this statement. Now we can prove the main features of the functions q_n.

Lemma 4.40. *For every n:*

- (*i*) *The function q_n is of class $\mathcal{C}^1([0,1])$.*
- (*ii*) *The restrictions $q_n|_{[0,\tau_n^-)}$ and $q_n|_{(\tau_n^+,1]}$ are $\mathcal{C}^2$ solutions of*

$$\widetilde{\omega}_n^2 q_n''(\tau) = \nabla_{q_n}\left(V_\epsilon^j(q_n^2(\tau)+c_j)|q_n(\tau)|^2\right) - 2q_n(\tau).$$

- (*iii*) *The energy of q_n is constant in $[0,1]$:*

$$\frac{1}{2}|q_n'(\tau)|^2 - \frac{1}{\widetilde{\omega}_n^2}\left(V_\epsilon^j(q_n^2(\tau)+c_j)-1\right)|q_n(\tau)|^2 = \frac{m_j}{\widetilde{\omega}_n^2} \qquad \forall \tau \in [0,1].$$

- (*iv*) *The variation of the angle on the constraint tends to 0 for $n \to \infty$:*

$$\lim_{n\to\infty} |\sigma_n(\tau_n^+) - \sigma_n(\tau_n^-)| = 0.$$

- (*v*) *The time interval on the constraint tends to 0 for $n \to \infty$:*

$$\lim_{n\to\infty} (\tau_n^+ - \tau_n^-) = 0.$$

Proof. The point (*i*) is obvious, the points (*ii*) and (*iii*) are consequence of the variational property of q_n, Lemma 4.37.
(*iv*) We can use the results already obtained in the step 3) (recall in particular the expression of u_n in polar coordinates, the definition of the sequence (v_n) and the expression of v_n in polar coordinates, the equations (4.27) and (4.28)).
The angle of the function q_n with respect to the origin is exactly half of the angle of u_n with respect to c_j. Hence we can we prove that

$$\lim_{n\to\infty} |\phi_n(t_n^+) - \phi_n(t_n^-)| = 0.$$

or equivalently

$$\lim_{n\to\infty} |\bar{\phi}_n(s_n^+) - \bar{\phi}_n(s_n^-)| = |\phi^+ - \phi^-| = 0.$$

From (4.27) and (4.28) we get

$$2\pi + |\phi^+ - \phi^-| \le 2\pi \Leftrightarrow |\phi^+ - \phi^-| = 0.$$

Recall that in the proof of (4.27) and (4.28) we supposed (it is not restrictive) the angle ϕ increasing. This is why there was no absolute value.
(*v*) It is a consequence of the same property for u_n, Proposition 4.32:

$$\begin{aligned}\tau_n^+ - \tau_n^- = \int_{\tau_n^-}^{\tau_n^+} d\tau = \int_{t_n^-}^{t_n^+} \frac{dt}{S_n|q_n(\tau(t))|^2} = \frac{t_n^+ - t_n^-}{S_n\rho_n} \\ = \frac{O(\rho_n^{\frac{2+\alpha}{2}})}{S_n\rho_n} \simeq \frac{\rho_n^{\frac{\alpha}{2}}}{S_n} \to 0\end{aligned}$$

for $n \to \infty$. □

Proposition 4.41. *The path $\widetilde{q}$ is a classical solution of*

$$\widetilde{\Omega}^2\widetilde{q}''(\tau) = \nabla_{\widetilde{q}}\left(V_\epsilon^j(\widetilde{q}^{\,2}(\tau)+c_j)|\widetilde{q}^{\,2}(\tau)|\right) - 2\widetilde{q}(\tau) \qquad \forall \tau \in [0,1]. \tag{4.30}$$

Proof. The point (v) of the previous lemma implies that the sequences (τ_n^-) and (τ_n^+) converge to $\tau_0 \in (0,1)$ such that $\widetilde{q}(\tau_0) = 0$, which corresponds to the unique collision time $t_0 \in (c,d)$ of $\widetilde{u}$. Since every q_n is $\mathcal{C}^1$, the vectors $q_n(\tau)$ is tangent to the boundary $\{q \in \mathbb{C} : |q| = \sqrt{\rho_n}\}$ for every $\tau \in [\tau_n^-, \tau_n^+]$. Moreover, the variation of the angle on the constraint tends to 0, so that

$$\lim_{\tau\to\tau_0^-}\widetilde{q}'(\tau) = \lim_{\tau\to\tau_0^+}\widetilde{q}'(\tau),$$

i.e. $\widetilde{q}$ passes trough the origin without any change of direction. We know that q_n uniformly converges to $\widetilde{q}$ over $[0,1]$, and it is not difficult to see that the restrictions $q_n|_{[0,\tau_n^-)}$ and $q_n|_{(\tau_n^+,1]}$ converge to $\widetilde{q}$ in the $\mathcal{C}^1$-topology (follow the proof of Proposition 4.34). Next, the minimality of q_n, the coercivity and the weak lower semi-continuity of $\widetilde{M}$ imply that $\widetilde{q}$ is a local minimizer of $\widetilde{M}$ itself. As a consequence it is a weak (and, by regularity, strong) solution of

$$\widetilde{\Omega}^2\widetilde{q}'' = \nabla_{\widetilde{q}}\left(V_\epsilon^j(\widetilde{q}^2+c_j)|\widetilde{q}^2|\right) - 2\widetilde{q}. \qquad \square$$

Conclusion of the proof of Proposition 4.35. Let us consider the functions

$$\widetilde{q}_1(\tau) = \widetilde{q}(\tau_0+\tau), \qquad \widetilde{q}_2(\tau) = -\widetilde{q}(\tau_0-\tau).$$

They are both solutions of (4.30) (as far as q^2 is concerned, pay attention to the change of sign in the transformation $\nabla_{\widetilde{q}} \rightsquigarrow \nabla_{\widetilde{q}^2}$) with the same initial values. Thanks to the regularity of (4.30), the uniqueness of the solution for the Cauchy's problem and the definition of the Levi-Civita transform gives

$$\widetilde{q}(\tau_0+\tau) = -\widetilde{q}(\tau_0-\tau) \Rightarrow \widetilde{u}(t_0+t) = \widetilde{u}(t_0-t):$$

if the function $\widetilde{u}$ has a collision, then necessarily bounce against one centre and comes back along the same trajectory until the point $p_1 = p_2$. Letting $x(t) = \widetilde{u}(\Omega t)$ for $t \in [c/\Omega, d/\Omega]$, the function can be uniquely extended over $[0, 1/\Omega]$ as an ejection-collision solution of problem (1.4) connecting x_1 and $x_2 = x_1$ (this uniqueness is a consequence of the uniqueness of the solutions for smooth Cauchy's problem. $\square$

We end this section with some remarks about our peculiar use of the Levi-Civita regularization.

Remark 4.42. We proved that, if the minimum of the restriction of M over K_l is achieved over a collision path, then we can find an ejection-collision minimizer in the same class. To do this, we built a minimizing sequence and then we passed to the limit in the Levi-Civita space. Thanks to the regularity of the transformed problem, we obtained an equation satisfied by the limit, and this implied the collision-ejection condition. Actually, the same procedure works if we consider a collision minimizer $u \in K_l^{p_1p_2}([0,1])$ which is a uniform limit of a sequence $u_n \in K_l^{p_1^np_2^n}([0,1])$ (of course, necessarily $p_1^n \to p_1$ and $p_2^n \to p_2$ when $n \to \infty$), where u_n is

- a collisions-free minimizer for M if $p_1^n \neq p_2^n$.
- a collisions-free minimizer for M if $p_1 = p_2$ and the minimum of M over $K_l^{p_1^np_2^n}$ is achieved over collisions-free paths.
- an ejection-collision minimizer for M if $p_1^n = p_2^n$ is achieved by a collision path.

Up to a subsequence, because of the uniform convergence, we can assume that either u_n is collisions-free for every $n \geq n_0$, or u_n is an ejection-collision solution for every $n \geq n_0$. In this latter case, the uniform convergence suffices to imply that u is an ejection collision path (reasoning as in step 5)).

On the Levi-Civita transform 4.43. As clearly explained in [17], the N-center problem admits a global Levi-Civita regularization. It consists in extending the pullback of the Jacobi metric on the concrete Riemann surface

$$\mathcal{R} = \left\{(u,Q) \,:\, Q^2 = \prod_{j=1}^{N}(u-c_j)\right\}$$

to a smooth metric. The projection from $\mathcal{R} \mapsto \mathbb{C}$ on the first factor is a branched covering of $\mathbb{C}$ whose branch points $C_j = (c_j, 0)$ are of order one and project on the centers $\{c_j\}$. The Riemann surface $\tilde{\mathcal{R}} = \mathcal{R} \setminus \{C_j\}$ doubly covers the configuration space $\mathbb{C} \setminus \{c_j\}$; moreover, there is a unique way of lifting the Jacobi metric to $\tilde{\mathcal{R}}$ and this extend in an unique way to a smooth metric on $\mathcal{R}$. Geodesics on $\mathcal{R}$ can be classified according with the fundamental group $\pi_1(\mathcal{R})$, which is known to be isomorphic to the free group on $N-1$ generators. The main

reason why we choose to use the local L-C transform is that we want to keep track of the topology of the true configuration space, and specially, of the number of self intersections and the rotation vectors with respect to the centres. This is a common feature of all the cases $\alpha \in [1,2)$.

This explains why we are lead to swinging back and forth from the configuration space to the Riemann surface.

Of course, also the local Levi-Civita transform induces a regularization of the flow associated with the first order system (1.3). Indeed, let us consider an ejection-collision solution $\widehat{y}$ of (1.4) starting from $p_0 \in \partial B_R(0)$, coming from an ejection-collision minimizer $\widehat{u} \in K_l^{p_0 p_0}([0,1])$. The Levi-Civita transform $\widehat{q}$ of $\widehat{u}$ is a regular solution of (4.30). Let us define the reparametrization $\widehat{\mathfrak{q}}(\tau) := \widehat{q}(\widetilde{\Omega}\tau)$; it is a regular solution of

$$\mathfrak{q}''(\tau) = \nabla_{\mathfrak{q}} \left(V_\epsilon^j(\mathfrak{q}(\tau)^2 + c_j)|\mathfrak{q}(\tau)|^2 \right) - 2\mathfrak{q}(\tau) \tag{4.31}$$

with energy m_j, starting from $\widehat{\mathfrak{x}}_0 \in \Lambda(x_0)$ and arriving to $\widetilde{\mathfrak{x}}_0 \in \Lambda(x_0)$, with $\widehat{\mathfrak{x}}_0 \neq \widehat{\mathfrak{x}}_0$. Now let us consider a collisions-free solution y_l of problem (1.4), with initial data $(p_1, \dot{x}_l(0))$ close to the initial data of $\widehat{y}$. This solution comes from a collisions-free minimizer $u_l \in K_l^{p_1 p_2}([0,1])$ of M, for some $p_2 \in \partial B_R(0)$ (see Remark 4.2). Even in this case we can consider the Levi-Civita transform $\Lambda(u_l)$ (centred in c_j), given by

$$u_l(t) = q_l^2(\tau(t)) + c_j$$
$$dt = S|q_l(\tau)|^2 \, d\tau$$
$$S = \int_0^1 \frac{dt}{|u_l(t) - c_j|}.$$

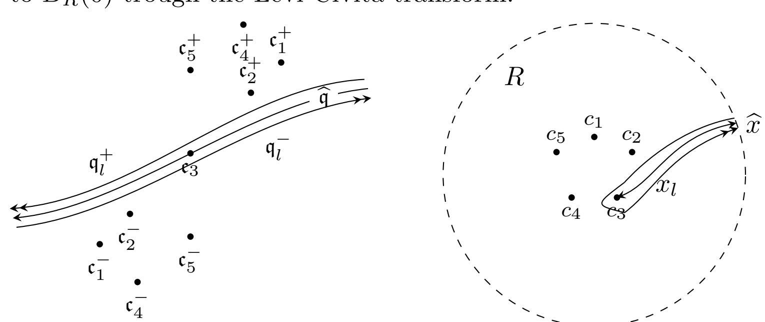
Each component $q_l \in \Lambda(u_l)$ is a local minimizer of $\widetilde{M}$ at a positive level. Setting,

$$\omega_l^2 := \frac{\int_0^1 \left[m_j + \left(V_\epsilon^j(q_l^2 + c_j) - 1 \right) |q_l|^2 \right]}{\frac{1}{2}\int_0^1 |q_l|^2},$$

we infer

$$\omega_q^2 q_l''(\tau) = \nabla_q \left(V_\epsilon^j(q_l(\tau)^2 + c_j)|q_l(\tau)|^2 \right) - 2q_l(\tau) \qquad \forall \tau \in [0,1]$$
$$\frac{1}{2}|q_l'(\tau)|^2 - \frac{1}{\widetilde{\omega}_n^2}\left(V_\epsilon^j(q_l^2(\tau) + c_j) - 1 \right)|q_l(\tau)|^2 = \frac{m_j}{\widetilde{\omega}_n^2} \qquad \forall \tau \in [0,1].$$

The reparametrization $\mathfrak{q}_l(\tau) = q_l(\omega_l \tau)$ is a solution of equation (4.31) with energy m_j and initial data close to those of $\widehat{\mathfrak{q}}$. This is a smooth equation, hence the continuous dependence of the solutions holds true: since the initial values of $\mathfrak{q}_l$ and of $\widehat{\mathfrak{q}}$ are close together, these solutions stays close together in a right neighbourhood of 0. In particular it is not difficult to see that this continuous dependence holds true if the solutions stays in the set which corresponds to $B_R(0)$ trough the Levi-Civita transform.



The picture represents a comparison between the Levi-Civita space (on the left), centred in $\mathfrak{c}_3 = \Lambda(c_3)$, and the configuration space (on the right) of the true N-centre problem. We have the ejection-collision solution $\widehat{x}$, with its collision with c_3. In the Levi-Civita space, the corresponding path $\widehat{\mathfrak{q}}$ solves the regular differential equation (4.31) (we fix an orientation of this solution given by the arrow). If we take one solution $\mathfrak{q}_l^+$ with similar initial data, we can apply the continuous dependence theorem: hence $\widehat{\mathfrak{q}}$ is close (in the uniform topology) to $\mathfrak{q}_l$. If we had chosen the inverse orientation for $\widehat{\mathfrak{q}}$, we would got the $\mathfrak{q}_l^-$. Coming back to the physical space, this means that if we take a solution with initial data close to those of a collision-ejection one, a continuous dependence exists (despite the lack of regularity of the potential!).

Let us note that, with the exception of c_3, each points of $\mathbb{R}^2$ corresponds to two points of the Levi-Civita space.

For instance $\mathfrak{c}_i^\pm \in \Lambda(c_i)$ for $i = 1, 2, 4, 5$. This is due to the fact that in the Levi-Civita space two points which are poles apart are identified when we come back to the physical space. Therefore, our ejection-collision solution correspond to a path crossing the origin and showing a central symmetry, connecting two points which are identified in $p_0 \in \mathbb{R}^2$. We could choose both the orientations for $\widehat{\mathfrak{q}}$, and the identification would give the same path in the physical space.
Let us also note that the angles with respect to the point c_3 in the physical space are cut by half in the Levi Civita one.

5. A finite dimensional reduction

In this section we glue the paths alternating outer and inner arcs in order to construct periodic orbits of the N-centre problem on the whole plane. Thus, our building blocks will be the fixed end trajectories found in the discussions of sections 3 and 4. In order to obtain smooth junctions, we are going to use a variational argument. To do this, we need to know that the time interval of each of them is conveniently bounded. This is the object of the following lemmas. We recall that ϵ_2, ϵ_3 have been introduced in Theorem 3.1 and Theorem 4.12 respectively

Lemma 5.1. *Let $0 < \epsilon < \min\{\epsilon_2, \epsilon_3\}$, let $p_0, p_1 \in \partial B_R(0)$, $|p_1 - p_0| < \delta$; let $y_{ext}(\cdot\,; p_0, p_1; \epsilon)$ the "exterior" solution of (1.4), found in Theorem 3.1; let $[0, T_{ext}(p_0, p_1; \epsilon)]$ its domain. Then there exist $C_1, C_2 > 0$ such that*

$$C_1 \le T_{ext}(p_0, p_1; \epsilon) \le C_2 \qquad \forall (p_0, p_1) \in (\partial B_R(0))^2 .$$

Proof. It is a straightforward consequence of the continuous dependence of the solutions by initial data and of the construction of $y_{\text{ext}}(\cdot\,; p_0, p_1; \epsilon)$ as a perturbed solution. □

Lemma 5.2. *Let $0 < \epsilon < \min\{\epsilon_2, \epsilon_3\}$, let $p_1, p_2 \in \partial B_R(0)$; let $y_{P_j}(\cdot\,; p_1, p_2; \epsilon)$ be a solution of (1.4), coming from a minimizer $u_{P_j}(\cdot\,; p_1, p_2; \epsilon) \in K_{P_j}^{p_1 p_2}([0, 1])$ of M, for some $P_j \in \mathcal{P}$; let $[0, T_{P_j}(p_1, p_2; \epsilon)]$ be its domain. Then there exist $C_3, C_4 > 0$ such that*

$$C_3 \le T_{P_j}(p_1, p_2; \epsilon) \le C_4 \qquad \forall (p_1, p_2) \in (\partial B_R(0))^2 ,\ \forall P_j \in \mathcal{P}.$$

Proof. Letting $T_{P_j}(p_1, p_2; \epsilon) = 1/\omega_{P_j}(p_1, p_2; \epsilon)$, we recall that

$$\omega_{P_j}(p_1, p_2; \epsilon) = \frac{\int_0^1 \left(V_\epsilon(u_{P_j}(t; p_1, p_2; \epsilon)) - 1\right)\, dt}{\int_0^1 |\dot{u}_{P_j}(t; p_1, p_2; \epsilon)|^2\, dt}.$$

Therefore we can prove that there exist $C_3, C_4 > 0$ such that

$$\frac{1}{C_4} \le \omega_{P_j}(p_1, p_2; \epsilon) \le \frac{1}{C_3} \qquad \forall (p_1, p_2) \in (\partial B_R(0))^2 ,\ \forall P_j \in \mathcal{P}.$$

Since $\mathcal{P}$ is a discrete and finite set, we can fix $P_j \in \mathcal{P}$ and apply the same reasoning for every j. Let us fix $\widetilde{p}_1, \widetilde{p}_2 \in \partial B_R(0)$. There exist $\widetilde{u}_* \in \widehat{K}_{P_j}^{\widetilde{p}_1 \widetilde{p}_2}([0, 1])$ and $C, \mu > 0$ such that

- $|\dot{\widetilde{u}}_*| = C$ for every $t \in [0, 1]$.
- $|\widetilde{u}_*(t) - c_k| \ge \mu$ for every $t \in [0, 1]$, for every $k = 1 \ldots, \ldots N$.

There holds

$$M(\widetilde{u}_*) = \frac{1}{2}\int_0^1 |\dot{\widetilde{u}}_*|^2 \int_0^1 (V_\epsilon(\widetilde{u}_*) - 1) = \frac{C^2}{2}\int_0^1 \left(\sum_{k=1}^N \frac{m_k}{\alpha|\widetilde{u}_* - c_k|^\alpha} + h\right) \le \frac{C^2}{2}\left(\frac{M}{\alpha\mu^\alpha} + h\right) =: C_5,$$

with $C_5 > 0$ since the ball of radius R is a subset of $\{V_\epsilon > 1\}$. Also, for every $u \in \bigcup_{p_1, p_2 \in \partial B_R(0)} K_{P_j}^{p_1 p_2}([0, 1])$,

$$\int_0^1 (V_\epsilon(u) - 1) \ge \frac{M}{\alpha\left(\bar{R} + \max_{1 \le k \le N} |c_k|\right)} + h =: C_6 \tag{5.1}$$

with $C_4 > 0$ for our choice of R. For a minimizer $\widetilde{u} = \widetilde{u}_{P_j}(\cdot\,; \widetilde{p}_1, \widetilde{p}_2; \epsilon) \in K_{P_j}^{\widetilde{p}_1 \widetilde{p}_2}([0, 1])$, one has

$$M(\widetilde{u}) = \frac{1}{2}\int_0^1 |\dot{\widetilde{u}}|^2 \int_0^1 (V_\epsilon(\widetilde{u}) - 1) \le M(\widetilde{u}_*),$$

which together with (5.1) gives

$$\int_0^1 |\dot{\widetilde{u}}|^2 \leq \frac{2C_5}{C_6}. \tag{5.2}$$

Starting from this bound for one single minimizer, it is not difficult to obtain a uniform bound (with respect to the ends) for every minimizers. Indeed if $(p_1, p_2) \neq (\widetilde{p}_1, \widetilde{p}_2)$, we consider the path

$$\widehat{u}_*(t) := \begin{cases} \sigma_R(t; p_1, \widetilde{p}_1) & t \in [0, 1/3] \\ \widetilde{u}_*(3t-1) & t \in (1/3, 2/3] \\ \sigma_R(t; \widetilde{p}_2, p_2) & t \in (2/3, 1], \end{cases}$$

where, for $p_*, p_{**} \in \partial B_R(0)$, $\sigma_R(\cdot\,; p_*, p_{**})$ is the shorter (in the Euclidean metric) arc of $\partial B_R(0)$ connecting p_* and p_{**} with constant angular velocity. As far as the angular velocity is concerned, it is easy to see that it is uniformly bounded with respect to p_*, p_{**}. This, together with the assumptions on $\widetilde{u}_*$, implies that also the velocity of $\widehat{u}_*$ is bounded in $[0, 1]$, and

$$M(\widehat{u}_*) \leq \frac{C^2}{2} \int_0^1 (V_\epsilon(\widehat{u}_*) - 1) = C + 3C \int_0^1 (V_\epsilon(\widetilde{u}_*) - 1) =: C_7.$$

This (positive) constant does not depend on the ends p_1 and p_2, so that for the family of the minimizers there holds

$$M(u_{P_j}(\cdot\,; p_1, p_2; \epsilon)) \leq C_7 \qquad \forall p_1, p_2 \in \partial B_R(0). \tag{5.3}$$

Collecting (5.1) and (5.3) we obtain

$$\int_0^1 |\dot{u}_{P_j}(\cdot\,; p_1, p_2; \epsilon)|^2 \leq \frac{2C_7}{C_6} =: C_8 \qquad \forall p_1, p_2 \in \partial B_R(0). \tag{5.4}$$

A few more observations: as we have already repeated many times, the paths in $\bigcup_{p_1,p_2 \in \partial B_R(0)} K_{P_j}^{p_1 p_2}([0, 1])$ are uniformly non-constant since they have to cover at least a distance $R - \epsilon > 0$. Thus there exists $C_9 > 0$ such that

$$\|\dot{u}\|_2^2 \geq C_9 \qquad \forall u \in \bigcup_{p_1,p_2 \in \partial B_{\bar{R}}(0)} K_{P_j}^{p_1 p_2}([0, 1]). \tag{5.5}$$

From (5.3) and (5.5) it follows

$$\int_0^1 \big(V_\epsilon(u_{P_j}(\cdot\,; p_1, p_2; \epsilon)) - 1\big) \leq \frac{C_4}{C_6} =: C_{10} \qquad \forall p_1, p_2 \in \partial B_R(0). \tag{5.6}$$

Collecting (5.1), (5.4), (5.5), (5.6) we obtain

$$C_9 \leq \inf_{p_1,p_2 \in \partial B_R(0)} \|\dot{u}_{P_j}(\cdot\,; p_1, p_2; \epsilon)\|_2^2 \leq \sup_{p_1,p_2 \in \partial B_R(0)} \|\dot{u}_{P_j}(\cdot\,; p_1, p_2; \epsilon)\|_2^2 \leq C_8$$

and

$$C_6 \leq \inf_{p_1,p_2 \in \partial B_R(0)} \int_0^1 \big(V_\epsilon(u_{P_j}(\cdot\,; p_1, p_2; \epsilon)) - 1\big) \leq \sup_{p_1,p_2 \in \partial B_R(0)} \int_0^1 \big(V_\epsilon(u_{P_j}(\cdot\,; p_1, p_2; \epsilon)) - 1\big) \leq C_{10}.$$

The assertion is now an immediate consequence of the definition of $\omega_{P_j}(p_1, p_2; \epsilon)$. □

For $0 < \epsilon < \min\{\epsilon_2, \epsilon_3\}$, $n \in \mathbb{N}$, let us fix a finite sequence of partitions $(P_{k_1}, P_{k_2}, \ldots, P_{k_n}) \in \mathcal{P}^n$. We define

$$D = \Big\{(p_0, \ldots, p_{2n}) \in (\partial B_R(0))^{2n+1} : |p_{2j+1} - p_{2j}| \leq \delta \ \text{ for } j = 0, \ldots, n-1,\ p_{2n} = p_0\Big\},$$

where δ has been introduced in Theorem 3.1. Let $(p_0, \ldots, p_{2n}) \in D$. For every $j \in \{0, \ldots, n-1\}$, we can apply Theorem 3.1 to obtain the uniquely determined path

$$y_{2j}(t) := y_{\text{ext}}(t; p_{2j}, p_{2j+1}; \epsilon) = r_{\text{ext}}(t; x_{2j}, x_{2j}; \epsilon) \exp\{i\theta_{\text{ext}}(t; x_{2j}, x_{2j}; \epsilon)\} \qquad t \in [0, T_{2j}], \tag{5.7}$$

where $T_{2j} := T_{\text{ext}}(p_{2j}, p_{2j+1}; \epsilon)$. Namely

$$\begin{cases} \ddot{y}_{2j}(t) = \nabla V_\epsilon(y_{2j}(t)) & t \in [0, T_{2j}], \\ \frac{1}{2}|\dot{y}_{2j}(t)|^2 - V_\epsilon(y_{2j}(t)) = -1 & t \in [0, T_{2j}], \\ |y_{2j}(t)| > R & t \in (0, T_{2j}), \\ y_{2j}(0) = p_{2j}, \quad y_{2j}(T_{2j}) = p_{2j+1}. \end{cases}$$

We recall that y_{2j} depends on p_{2j} and p_{2j+1} in a $\mathcal{C}^1$ manner. On the other hand, for every $j = 0, \dots, n-1$, we can find trough Corollary 4.14 a path

(5.8)
$$y_{2j+1}(t) := y_{P_{k_{j+1}}}(t; p_{2j+1}, p_{2j+2}; \epsilon) = r_{P_{k_{j+1}}}(t; p_{2j+1}, p_{2j+2}; \epsilon) \exp\{i\theta_{P_{k_{j+1}}}(t; p_{2j+1}, p_{2j+2}\epsilon)\} \qquad t \in [0, T_{2j+1}],$$

where $T_{2j+1} := T_{P_{k_{j+1}}}(p_{2j+1}, p_{2j+2}; \epsilon)$. Namely y_{2j+1} is a path of $K_{P_{k_{j+1}}}$ such that

$$\begin{cases} \ddot{y}_{2j+1}(t) = \nabla V_\epsilon(y_{2j+1}(t)) & t \in [0, T_{2j+1}]\,, \\ \frac{1}{2}|\dot{y}_{2j+1}(t)|^2 - V_\epsilon(y_{2j+1}(t)) = -1 & t \in [0, T_{2j+1}]\,, \\ |y_{2j+1}(t)| < R & t \in (0, T_{2j+1})\,, \\ y_{2j+1}(0) = p_{2j+1}, \quad y_{2j+1}(T_{2j+1}) = p_{2j+2}. \end{cases}$$

We know that if $\alpha \neq 1$ or $\alpha = 1$ and $p_{2j+1} \neq p_{2j+2}$ then y_{2j+1} is collisions-free, while if $\alpha = 1$ and $p_{2j+1} = p_{2j+2}$ y_{2j+1} can be an ejection-collision solution. Due to the invariance under reparametrizations, y_{2j+1} is a local minimizer of the functional $L\left([0, T_{2j+1}]\,; \cdot\right)$.

Let us set $\mathfrak{T}_k := \sum_{j=0}^{k} T_j$, $k = 0, \dots, 2n-1$. We define

$$\gamma_{(p_0,\dots,p_{2n})}(s) := \begin{cases} y_0(s) & s \in [0, \mathfrak{T}_0] \\ y_1(s - \mathfrak{T}_0) & s \in [\mathfrak{T}_0, \mathfrak{T}_1] \\ \vdots & \\ y_{2n-2}\left(s - \mathfrak{T}_{2n-3}\right) & s \in [\mathfrak{T}_{2n-3}, \mathfrak{T}_{2n-2}] \\ y_{2n-1}\left(s - \mathfrak{T}_{2n-2}\right) & s \in [\mathfrak{T}_{2n-2}, \mathfrak{T}_{2n-1}]. \end{cases}$$

The function $\gamma_{(p_0,\dots,p_{2n})}$ is a piecewise differentiable $\mathfrak{T}_{2n-1}$-periodic function; to be precise, if $\alpha \in (1,2)$ it is a classical solution of the N-centre problem (1.4) with energy -1 in $[0, \mathfrak{T}_{2n-1}] \setminus \{0, \mathfrak{T}_0, \dots, \mathfrak{T}_{2n-1}\}$; in general, it is not $\mathcal{C}^1$ in $\{0, \mathfrak{T}_0, \dots, \mathfrak{T}_{2n-1}\}$, but the right and left limits in these points are finite, so that it is in H^1. If $\alpha = 1$ it is possible also that $\gamma_{(p_0,\dots,p_{2n})}$ has a finite number of collisions. Let us observe that, thanks to Lemmas 5.1 and 5.2, we are sure that the time interval of $\gamma_{(p_0,\dots,p_{2n})}$ is bounded above and bounded below by a positive constant for every $(p_0, \dots, p_{2n})$, so that the period is neither trivial, nor infinite.

Finally, we introduce the function $F_{((P_{k_1},\dots,P_{k_n});\epsilon)} : D \to \mathbb{R}$ defined by

$$F(p_0, \dots, p_{2n}) := L\left([0, \mathfrak{T}_{2n-1}]; \gamma_{(p_0,\dots,p_{2n})}\right) = \sum_{j=0}^{2n-1} \int_0^{T_j} \sqrt{\left(V(y_j) - 1\right)|\dot{y}_j|^2} = \sum_{j=0}^{2n-1} L\left([0, T_j]; y_j\right).$$

We point out that F depends on $(P_{k_1}, \dots, P_{k_n})$ and ϵ trough the dependence on these quantity of $\{y_j\}$. Since we fixed $(P_{k_1}, \dots, P_{k_n})$ and ϵ, we will omit subscripts and instead of $F_{((P_{k_1},\dots,P_{k_n});\epsilon)}$ we will simply write F. The main goal of this section is to prove the following theorem.

Theorem 5.3. *There exists $(p_0, \dots, p_{2n}) \in D$ which minimizes F. There exists $\bar{\epsilon} > 0$ such that, if $\epsilon \in (0, \bar{\epsilon})$, then the associated function $\gamma_{(p_0,\dots,p_{2n})}$ is a periodic solution of the N-centre problem* (1.4) *with energy -1. The value $\bar{\epsilon}$ depends neither on n, nor on $(P_{k_1}, \dots, P_{k_n}) \in \mathcal{P}^n$. Moreover:*

- (i) *if $\alpha \in (1,2)$ then $\gamma_{(p_0,\dots,p_{2n})}$ is collisions-free.*
- (ii) *if $\alpha = 1$ there are three possibilities:*
  - $a)$ *$\gamma_{(p_0,\dots,p_{2n})}$ is collisions-free.*

b) $\gamma_{(p_0,\dots,p_{2n})}$ *has a collision with one centre* c_j*, covers a certain trajectory, rebounds against a centre* c_k *(it can occur* $c_j = c_k$*) and come back along the same trajectory. This is possible just if* n *is even and* $(P_{k_1},\dots,P_{k_n})$ *is equivalent to* $(P'_{k_1},\dots,P'_{k_n})$ *such that*

$$P'_{k_1} = Q_j \in \mathcal{P}_1, \quad P'_{j_{n/2+1}} = Q_k \in \mathcal{P}_1 \quad \text{and (if } n > 2\text{)}$$
$$P'_{k_n} = P'_{k_2},\ P'_{k_{n-1}} = P'_{k_3},\ \dots,\ P'_{k_{n/2+2}} = P'_{k_{n/2}}.$$

c) $\gamma_{(p_0,\dots,p_{2n})}$ *has a collision iwith one centre* c_j*, covers a certain path, "rebounds" against the surface* $\left\{x \in \mathbb{R}^2 : V_\epsilon(x) = 1\right\}$ *with null velocity and come back along the same trajectory. This is possible just if* n *is odd and* $(P_{k_1},\dots,P_{k_n})$ *is equivalent to* $(P'_{k_1},\dots,P'_{k_n})$ *such that*

$$P'_{k_1} = Q_j \in \mathcal{P}_1 \quad \text{and (if } n > 1\text{)}$$
$$P'_{k_n} = P'_{k_2},\ P'_{k_{n-1}} = P'_{k_3},\ \dots,\ P'_{k_{(n+1)/2+1}} = P_{k_{(n+1)/2}}.$$

Remark 5.4. Theorem 1.1 is a trivial consequence of this result, see also Remark 2.3: given $0 < \epsilon < \bar{\epsilon}$, for every $n \in \mathbb{N}$ and for every $(P_{k_1},\dots,P_{k_n}) \in \mathcal{P}^n$ we found a periodic solution $\gamma_{(p_0,\dots,p_{2n})}$ of (1.4), whose behaviour is determined by $(P_{k_1},\dots,P_{k_n})$. Let us set $\bar{h} = -\zeta(\bar{\epsilon})$. Now, given $\bar{h} < h < 0$, for every $n \in \mathbb{N}$ and $(P_{k_1},\dots,P_{k_n}) \in \mathcal{P}^n$ we obtain a periodic solution $x_{(P_{k_1},\dots,P_{k_n})}$ of the problem (1.1) with energy h via Proposition 2.1. As we pointed out at the end of section 2, the behaviour of $x_{(P_{k_1},\dots,P_{k_n})}$ and of $\gamma_{(p_0,\dots,p_{2n})}$ is the same.

Proof. The proof requires several steps. It is essential to keep in mind the notations introduced above.

Step 1) Existence of a minimizer. The set D is compact since it is a closed subset of the compact set $(\partial B_R(0))^{2n+1}$. It remains to show that F is continuous. Let $((p_0^m,\dots,p_{2n}^m))$ a convergent sequence in D: $\lim_m (p_0^m,\dots,p_{2n}^m) = (p_0,\dots,p_{2n}) \in D$. Let us consider

$$F(p_0^m,\dots,p_{2n}^m) = \sum_{j=0}^{n-1} L\left([0,T_{2j}^m]; y_{2j}^m\right) + \sum_{j=0}^{n-1} L\left([0,T_{2j+1}^m]; y_{2j+1}^m\right).$$

Here y_{2j}^m (resp. y_{2j+1}^m) is defined as y_{2j} (resp. y_{2j+1}); it has boundary values p_{2j}^m, p_{2j+1}^m (resp. p_{2j+1}^m, p_{2j+2}^m), and domain $[0,T_{2j}^m]$ (resp. $[0,T_{2j+1}^m]$).
The first sum is continuous in D, since the function y_{2j}^m depends in a differentiable way on its ends. As far as the second sum is concerned, we can consider the first addendum and repeat the reasoning for the others. For $x_*, x_{**} \in \partial B_R(0)$ we define $\sigma_R(\cdot\,; p_*, p_{**})$ as the shorter (in the Euclidean metric) arc of $\partial B_R(0)$ connecting p_* and p_{**}, parametrized in $[0,1]$. Obviously,

$$\forall \lambda > 0 \exists \varrho > 0 : |p_* - p_{**}| < \varrho \Rightarrow L\left([0,1]; \sigma_R\left(\cdot\,; p_*, p_{**}\right)\right) < \lambda.$$

Since y_1 minimizes L among the paths connecting p_1 and p_2 which separates the centres according to P_{k_1}, we have

$$L\left([0,T_1]; y_1\right) \le L\left([0,T_1^m]; y_1^m\right) + +L\left([0,1]; \sigma_R\left(\cdot\,; p_1^m, p_1\right)\right) + L\left([0,1]; \sigma_R\left(\cdot\,; p_2^m, p_2\right)\right). \tag{5.9}$$

Here we use the invariance of L under reparametrizations, so that it is possible to compare the values of L for functions defined over different time-intervals.
Analogously, the minimal property of y_1^m implies

$$L\left([0,T_1^m]; y_1^m\right) \le L\left([0,T_1]; y_1\right) + +L\left([0,1]; \sigma_R\left(\cdot\,; y_1^m, y_1\right)\right) + L\left([0,1]; \sigma_R\left(\cdot\,; y_2^m, y_2\right)\right). \tag{5.10}$$

Passing to the limit for $m \to \infty$ in (5.9) and (5.10), we get

$$\lim_{m\to\infty} L\left([0,T_1^m]; y_1^m\right) = L\left([0,T_1]; y_1\right).$$

Therefore F is continuous on D, and has a minimum.

Step 2) F has partial derivatives in D°. We point out that we are not proving that F is differentiable in D°. Let us fix $k \in \{0,\dots,2n\}$, $(p_0,\dots,p_{2n}) \in D^\circ$.

$$F(p_0,\dots,p_{2n}) = \sum_{j=0}^{n-1} L\left([0,T_{2j}]; y_{2j}\right) + \sum_{j=0}^{n-1} L\left([0,T_{2j+1}]; y_{2j+1}\right).$$

The first sum is differentiable with respect to p_k, since y_{2j} depends smoothly on its ends for every j. As far as the second sum, just one term depends on p_k. It is the same to consider k even or odd, so we define $k = 2j+1$

for some $j = 0, \dots, n-1$. We introduce a strongly convex neighbourhood U of the point p_{2j+1} with respect to the Jacobi's metric. We can assume that there exists $t_* \in (0, T_{2j+1}]$ such that

$$y_{2j+1}(t_*) \in (\partial U \cap y_{2j+1}([0, T_{2j+1}])) \quad \text{and} \quad t \in [0, t^*) \Rightarrow y_{2j+1}(t) \in U^\circ$$

There exists a unique minimizing geodesic $\widetilde{y}$ for the Jacobi metric, parametrized with respect to the arc length, connecting p_{2j+1} and $p_{2j+1}(t_*)$ in a certain time $\bar{t}$ and lying in U, which depends smoothly on its ends. For the uniqueness and the minimality of y_{2j+1}, this geodesic has to be a reparametrization of y_{2j+1}. Hence

$$L([0, \bar{t}]; \widetilde{y}) = L([0, t^*]; y_{2j+1}),$$

and the differentiability of the right side with respect to p_{2j+1} is a consequence of the differentiability of the left one. Now it is sufficient to note that

$$L([0, T_{2j+1}]; y_{2j+1}) = L([0, t^*]; y_{2j+1}) + L([t^*, T_{2j+1}]; y_{2j+1});$$

hence the left side is differentiable with respect to p_{2j+1}.

Step 3) Computation of the partial derivatives. Let us assume $k = 1$ to ease the notation (for the other k the computation is exactly the same). For $(p_0, \dots, p_{2n}) \in D^\circ$, there holds

$$\frac{\partial F}{\partial p_1}(p_0, \dots, p_{2n}) = \frac{\partial}{\partial p_1} L([0, T_0]; y_0) + \frac{\partial}{\partial p_1} L([0, T_1]; y_1). \tag{5.11}$$

We point out that this is a linear operator from the tangent space $T_{p_1}(\partial B_R(0))$ into $\mathbb{R}$. Let us consider the first term in the right side.

$$\begin{aligned}\frac{\partial}{\partial p_1} L([0, T_0]; y_0) &= dL([0, T_0]; y_0)\left[\frac{\partial y_0}{\partial p_1}\right] = \frac{1}{\sqrt{2}} \int_0^{T_0} \left[\left\langle \dot{y}_0, \frac{d}{dt}\frac{\partial y_0}{\partial p_1}\right\rangle + \left\langle \nabla V_\epsilon(y_0), \frac{\partial y_0}{\partial p_1}\right\rangle\right] \\ &= \frac{1}{\sqrt{2}} \int_0^{T_0} \left[\left\langle -\ddot{y}_0 + \nabla V_\epsilon(y_0), \frac{\partial y_0}{\partial p_1}\right\rangle\right] + \frac{1}{\sqrt{2}}\left[\left\langle \dot{y}_0(t), \frac{\partial y_0}{\partial p_1}(t)\right\rangle\right]_{t=0}^{t=T_0} = \frac{1}{\sqrt{2}}\left[\left\langle \dot{y}_0(t), \frac{\partial y_0}{\partial p_1}(t)\right\rangle\right]_{t=0}^{t=T_0}\end{aligned}$$

In the second equality we use the conservation of the energy for y_0, in the last one we use the fact that y_0 is a classical solution of the motion equation.

Every $\varphi \in T_{p_1}(\partial B_R(0))$ is of the form

$$\varphi = \beta'(0) \quad \text{for some } \beta : I \to \partial B_R(0) \text{ of class } \mathcal{C}^1,\ \beta(0) = p_1;$$

in the next step it will be useful to notice that, if $p_1 = R\exp\{i\theta_1\}$, then $T_{p_1}(\partial B_R(0))$ is spanned by $i\exp\{i\theta_1\}$. For $\varphi = \beta'(0) \in T_{p_1}(\partial B_R(0))$ there holds

$$\frac{\partial}{\partial p_1} y_0(0)[\beta'(0)] = \lim_{\lambda \to 0} \frac{y_{\text{ext}}(0; x_0, \beta(\lambda); \epsilon) - y_{\text{ext}}(0; x_0, x_1; \epsilon)}{\lambda} = 0$$

and

$$\frac{\partial}{\partial p_1} y_0(T_0)[\beta'(0)] = \lim_{\lambda \to 0} \frac{y_{\text{ext}}(T_{\text{ext}}(p_0, \beta(\lambda); \epsilon); p_0, \beta(\lambda); \epsilon) - y_0(T_0; p_0, p_1; \epsilon)}{\lambda} = \beta'(0),$$

where $y_{\text{ext}}(\cdot\,; p_0, \beta(l); \epsilon)$ is the exterior solution of (1.4) connecting p_0 and $\beta(\lambda)$ in time $T_{\text{ext}}(p_0, beta(\lambda); \epsilon)$ (recall that, by the proof of Theorem 3.1, we can find such a solution if $\beta(\lambda)$ is sufficiently close to p_0, even if it is not on $\partial B_R(0)$). Therefore, for every $\varphi \in T_{p_1}(\partial B_R(0))$

$$\frac{\partial}{\partial p_1} L([0, T_0]; y_0)[\varphi] = \frac{1}{\sqrt{2}} \langle \dot{y}_0(T_0), \varphi \rangle.$$

As far as the second term in the right side of the (5.11) is concerned, we can repeat the same computation obtaining

$$\frac{\partial}{\partial p_1} L([0, T_1]; y_1)[\varphi] = -\frac{1}{\sqrt{2}} \langle \dot{y}_1(0), \varphi \rangle \qquad \forall \varphi \in T_{p_1}(\partial B_R(0)).$$

Step 4) The minimizer is an inner point of D. Assume by contradiction that there exists $j \in \{0, \dots, n-1\}$ such that $|p_{2j} - p_{2j+1}| = \delta$. We can produce an explicit variation of p_{2j+1} such that F decreases along this variation, in contradiction with the minimality of $(p_0, \dots, p_{2n})$. It is not restrictive assume $j = 0$, the same argument applies for the other cases. In this step we use the notations (5.7) and (5.8) introduced above. The function $y_0(\cdot) = y_{\text{ext}}(\cdot\,; p_0, p_1; \epsilon)$ is a solution of

$$\begin{cases} \ddot{y}(t) = \nabla V_\epsilon(y(t)) \\ y(0) = p_0 = Re^{i\theta_0}, \quad \dot{y}(0) = \dot{r}_\epsilon e^{i\theta_0} + R\dot{\theta}_0 e^{i\theta_0}, \end{cases}$$

where

$$\dot{\theta}_0 = \dot{\theta}_0(p_0, p_1; \epsilon), \qquad \dot{r}_\epsilon = \dot{r}_\epsilon(\dot{\theta}_0),$$

and exists $T_0 = T_{\text{ext}}(p_0, p_1; \epsilon)$ such that $y(T_0; p_0, p_1; \epsilon) = p_1$ (see section 3).
As far as the function $y_1 = y_{P_{k_1}}(\cdot\,; p_1, p_2; \epsilon)$ is concerned, it solves

$$\begin{cases} \ddot{y}(t) = \nabla V_\epsilon(y(t)) \\ y(0) = p_1 = Re^{i\theta_1}, \quad v(T_1) = p_2 = Re^{i\theta_2} \\ y \in \widehat{K}_{P_j}\left([0, T_1]\right). \end{cases}$$

For $\epsilon = 0$ the angular momentum of the exterior solution is constant, so that

$$\dot{\theta}_0(p_0, p_1; \epsilon) = \dot{\theta}_{\text{ext}}(T_{\text{ext}}(p_0, p_1; 0); p_0, p_1; 0).$$

Let us set $\dot{\bar{\theta}}_0 := \dot{\theta}_0(p_0, p_1; 0)$, $\bar{T}_0 := T_{\text{ext}}(p_0, p_1; 0)$, and assume $\dot{\bar{\theta}}_0 > 0$ (the case $\dot{\bar{\theta}}_0 < 0$ is analogue). The continuous dependence of the solutions by vector field and initial data implies that

$$\forall \lambda > 0 \ \exists \epsilon_4 > 0 : \ 0 < \epsilon < \epsilon_4 \Rightarrow \left| \dot{\theta}_{\text{ext}}\left(T_{\text{ext}}(p_0, p_1; \epsilon); p_0, p_1; \epsilon\right) - \dot{\bar{\theta}}_0 \right| < \lambda.$$

With the choice $\lambda = \dot{\bar{\theta}}_0/2$ we get

$$0 < \frac{1}{2}\dot{\bar{\theta}}_0 < \dot{\theta}_{\text{ext}}\left(T_{\text{ext}}(p_0, p_1; \epsilon); p_0, p_1; \epsilon\right) < \frac{3}{2}\dot{\bar{\theta}}_0 \qquad \text{if } 0 < \epsilon < \epsilon_4. \tag{5.12}$$

Coming back to the function $y_{P_{k_1}}(\cdot\,; p_1, p_2; \epsilon)$, we define $S = S(p_1, p_2; \epsilon) \in \mathbb{R}^+$ by

$$t \in (0, S) \Rightarrow \frac{R}{2} < |y_1(t)| < R \text{ and } |y_1(S)| = R.$$

The energy integral makes this quantity uniformly bounded from below by a positive constant C, as function of ϵ.
Letting $\epsilon \to 0^+$ the centres collapse in the origin, so that for the angular momentum of $y_{P_{k_1}}(\cdot\,; p_1, p_2; \epsilon)$ it results

$$\mathfrak{C}_{y_{P_{k_1}}(\cdot\,; p_1, p_2; \epsilon)}(t) = o(\epsilon) \quad \text{for} \quad \epsilon \to 0^+,$$

uniformly in $[0, C]$ (recall Proposition 4.18). Consequently,

$$\forall \lambda > 0 \ \exists \epsilon_5 > 0 : 0 < \epsilon < \epsilon_5 \Rightarrow \left| \dot{\theta}_{P_{k_1}}\left(0; p_1, p_2; \epsilon\right) \right| < \lambda.$$

The choice $\lambda = \dot{\bar{\theta}}_0/3$ gives

$$\left| \dot{\theta}_{P_{k_1}}(0; p_1, p_2; \epsilon) \right| < \frac{1}{3}\dot{\bar{\theta}}_0 \qquad \text{if } 0 < \epsilon < \epsilon_5. \tag{5.13}$$

To conclude, we consider a variation $\varphi \in T_{p_1}(\partial B_R(0))$ of p_1 directed towards p_0 on $\partial B_R(0)$: since we are assuming $\dot{\bar{\theta}}_0 > 0$, this variation is a positive multiple of $-i \exp\{i\theta_1\}$. Collecting (5.12), (5.13) and the step 3, we obtain that if $0 < \epsilon < \min\{\epsilon_2, \epsilon_3, \epsilon_4, \epsilon_5\} =: \bar{\epsilon}$, then

$$\begin{aligned} \frac{\partial F}{\partial p_1}(p_0, \dots, p_{2n})[\varphi] &= \frac{CR}{\sqrt{2}} \left\langle \left(\dot{\theta}_{\text{ext}}\left(T_{\text{ext}}(p_0, p_1; \epsilon); p_0, p_1; \epsilon\right) - \dot{\theta}_{P_{k_1}}\left(0; p_1, p_2; \epsilon\right) \right) ie^{i\theta_1}, -ie^{i\theta_1} \right\rangle \\ &< \frac{CR}{\sqrt{2}} \left(\frac{\dot{\bar{\theta}}_0}{3} - \frac{\dot{\bar{\theta}}_0}{2} \right) < 0, \end{aligned}$$

against the minimality of $(p_0, \dots, p_{2n})$. We point out that $\bar{\epsilon}$ is independent on $n \in \mathbb{N}$ and on $(P_{k_1}, \dots, P_{k_n}) \in \mathcal{P}^n$.

Step 5) Regularity of the minimizers. Let $(p_0,\dots,p_{2n})\in D^\circ$ be a minimizer of F. Since $(p_0,\dots,p_{2n})$ is an inner point of D, the existence of the partial derivatives implies that

$$\frac{\partial F}{\partial p_k}(p_0,\dots,p_{2n})=0 \qquad \forall k=0,\dots,2n.$$

We consider for instance $k=2j+1$. From step 3) we know that for all $\varphi\in T_{p_k}(\partial B_R(0))$

$$\frac{1}{\sqrt{2}}\langle \dot{y}_{2j}\left(T_{2j}\right)-\dot{y}_{2j+1}\left(0\right),\varphi\rangle=0.$$

The tangent space $T_{p_k}(\partial B_R(0))$ is spanned by a single vector, $ie^{i\theta_{2j+1}}$. We deduce

$$|\dot{y}_{2j}\left(T_{2j}\right)|\cos\left(\widehat{\dot{y}_{2j}(T_{2j}),ie^{i\theta_{2j+1}}}\right)=|\dot{y}_{2j+1}\left(0\right)|\cos\left(\widehat{\dot{y}_{2j+1}(0),ie^{i\theta_{2j+1}}}\right)$$

Here $(\widehat{\dot{y}_{2j}(T_{2j}),ie^{i\theta_{2j+1}}})$, (resp. $(\widehat{\dot{y}_{2j+1}(0),ie^{i\theta_{2j+1}}})$) denotes the angle between the vectors $\dot{y}_{2j}(T_{2j})$ and $ie^{i\theta_{2j+1}}$ (resp. $\dot{y}_{2j+1}(0)$ and $ie^{i\theta_{2j+1}}$). As a consequence of the conservation of the energy

$$|\dot{y}_{2j}\left(T_{2j}\right)|=|\dot{y}_{2j+1}\left(0\right)| \tag{5.14}$$

so that $\cos\left(\widehat{\dot{y}_{2j}(T_{2j}),ie^{i\theta_{2j+1}}}\right)=\cos\left(\widehat{\dot{y}_{2j+1}(0),ie^{i\theta_{2j+1}}}\right)$. Both $\dot{y}_{2j}\left(T_{2j}\right)$ and $\dot{y}_{2j+1}\left(0\right)$ point towards the interior of $B_{R/2}(0)$, so that

$$\left(\widehat{\dot{y}_{2j}(T_{2j}),ie^{i\theta_{2j+1}}}\right)=\left(\widehat{\dot{y}_{2j+1}(0),ie^{i\theta_{2j+1}}}\right). \tag{5.15}$$

Collecting (5.14) and (5.15) we can conclude that $\dot{y}_{2j}(T_{2j})=\dot{y}_{2j+1}(0)$; hence

$$(p_0,\dots,p_{2n})\in D \text{ is a minimizer of } F\Rightarrow \gamma_{(p_0,\dots,p_{2n})}\in\mathcal{C}^1\left([0,\mathfrak{T}_{2n-1}]\right).$$

Step 6) Conclusion of the proof. Let $(p_0,\dots,p_{2n})$ be a minimizer of F in D°. If $\alpha\in(1,2)$ the function $\gamma_{(p_0,\dots,p_{2n})}$ is a classical solution of the N-centres problem with energy -1 in $[0,\mathfrak{T}_{2n-1}]\setminus\{0,\mathfrak{T}_0,\dots,\mathfrak{T}_{2n-1}\}$, and it is of class $\mathcal{C}^1$ in the interval $[0,\mathfrak{T}_{2n-1}]$. Since

$$\gamma_{(p_0,\dots,p_{2n})}(0)=\gamma_{(p_0,\dots,p_{2n})}(\mathfrak{T}_{2n-1}),\qquad \dot\gamma_{(p_0,\dots,p_{2n})}(0)=\dot\gamma_{(p_0,\dots,p_{2n})}(\mathfrak{T}_{2n-1}),$$

it can be defined over all $\mathbb{R}$ by periodicity. If we prove that it is of class $\mathcal{C}^2$, we can say that $\gamma_{(p_0,\dots,p_{2n})}$ is a classical periodic solution and the proof is complete. Let us fix $k=2j+1$, $j\in\{0,\dots,n-1\}$ (for k even the same reasoning applies). It results

$$\begin{aligned}\lim_{t\to\mathfrak{T}_{2j+1}^-}\ddot\gamma_{(p_0,\dots,p_{2n})}(t)=\lim_{t\to T_{2j+1}^-}\ddot{y}_{2j+1}(t)=\lim_{t\to T_{2j+1}^-}\nabla V(y_{2j+1}(t))=&\\ =\lim_{t\to0^+}\nabla V(y_{2j+2}(t))=\lim_{t\to0^+}\ddot{y}_{2j+2}(t)=\lim_{t\to\mathfrak{T}_{2j+1}^+}\ddot\gamma_{(p_0,\dots,p_{2n})}(t);&\end{aligned}$$

this completes the proof for $\alpha\in(1,2)$. If $\alpha=1$ it is possible that $\gamma_{(p_0,\dots,p_{2n})}$ is collisions-free; in such a case the same line of reasoning leads to alternative (ii)-$a)$ in Theorem 5.3. If a collision occur, we aim at showing that necessarily we are in cases (ii)-$b)$ or (ii)-$c)$. From Corollary 4.14, a necessary condition for the presence of collisions is the existence of $k_j\in\mathcal{P}_1$ for some $j\in\{1,\dots,n\}$; by possibly applying the right shift a number of times, it is not restrictive to assume that $j=1$. First of all we prove that $\gamma_{(p_0,\dots,p_{2n})}$ has to bounce again against a centre or against the surface $\{y\in\mathbb{R}^2:V_\epsilon(y)=1\}$. Let t^* its first collision time. Since v_1 is an ejection-collision trajectory, $\gamma_{(p_0,\dots,p_{2n})}$ has the same property:

$$\gamma_{(p_0,\dots,p_{2n-1})}(t^*+t)=\gamma_{(p_0,\dots,p_{2n-1})}(t^*-t)\qquad\forall t\in\mathbb{R}; \tag{5.16}$$

this is a consequence of the uniqueness of the solutions for the Cauchy's problem with initial point different from a singularity of the potential. On the other hand, since $\gamma_{(p_0,\dots,p_{2n-1})}$ has period $\mathfrak{T}_{2n-1}$, it has a reflection symmetry also with respect to $t^*+\mathfrak{T}_{2n-1}/2$. This second reflection can be smooth just if $\dot\gamma_{(p_0,\dots,p_{2n})}(t^*+\mathfrak{T}_{2n-1}/2)=0$, namely if $V_\epsilon\left(\gamma_{(p_0,\dots,p_{2n})}\left(t^*+\frac{\mathfrak{T}_{2n-1}}{2}\right)\right)=1$; otherwise $t^*+\mathfrak{T}_{2n-1}/2$ has to be another collision instant. In conclusion, we note that the reflection symmetry of the solution impose some symmetry restrictions on the sequence $(P_{k_1},\dots,P_{k_n})$, which we stated in Theorem 1.1. $\square$

6. Symbolic dynamics

In this section we fix $\alpha \in [1,2)$ and $h \in (\bar{h}, 0)$. Let us rewrite some partial results obtained for the normalized problem (energy -1 with parameter $\epsilon \in (0,\bar{\epsilon})$) in term of the "original" N-centre problem (to find solution of (1.1) with energy h). From Corollary 2.2 we detect a unique $\epsilon \in (0,\bar{\epsilon})$ such that $h = \zeta(\epsilon)$. In section 3 we found a solution $y_{\text{ext}}(\cdot\,; p_0, p_1; \epsilon)$ of (1.4) which stays outside $\partial B_R(0)$, and connects two points $p_0, p_1 \in \partial B_R(0)$ if their distance is smaller then δ. Via Proposition 2.1 we get a correspondent solution $x_{\text{ext}}(\cdot\,; x_0, x_1; h)$ for equation (1.1) with energy $h = \zeta(\epsilon)$, defined over an interval $[0, T_{\text{ext}}(x_0, x_1; h)]$. This solution connects $x_0, x_1 \in \partial B_{\bar{R}}$ close together (whose distance is smaller then $\bar{\delta}$), too, and stay outside $\partial B_{\bar{R}}(0)$. In section 4 we found a solution $y_{P_j}(\cdot\,; p_1, p_2; \epsilon)$ of (1.4) connecting $p_1, p_2 \in \partial B_R(0)$, which comes from a minimizer u of the Maupertuis' functional (with energy -1 and potential V_ϵ) in the class $K_{P_j}^{p_1 p_2}([0,1])$. Via Proposition 2.1 we get a correspondent solution $x_{P_j}(\cdot\,; x_1, x_2; h)$ for equation (1.1) with energy $h = \zeta(\epsilon)$, connecting $x_1, x_2 \in \partial B_{\bar{R}}(0)$, and defined over an interval $[0, T_{P_j}(x_1, x_2; h)]$. We set $T_{P_j}(x_1, x_2; h) = 1/\omega(x_1, x_2; h)$. As we mentioned in Remark 4.2, $x_{P_j}(\cdot\,; x_1, x_2; h)$ is a reparametrization of a critical point $\mathfrak{u}_{P_j}(\cdot\,; x_1, x_2; h)$ of the Maupertuis' functional (with energy h and with potential V) at a positive level. To be precise, since there is a correspondence between the space of the original problem (1.1) and the space of the normalized problem (1.4), the path $\mathfrak{u}_{P_j}(\cdot\,; x_1, x_2; h)$ is a minimizer of M_h in $\mathfrak{K}_{P_j}^{x_1 x_2}([0,1])$, which is the closure in the weak topology of H^1 of

$$\begin{aligned}\widehat{\mathfrak{K}}_{P_j}^{x_1 x_2}([0,1]) := \big\{ \mathfrak{v} \in H^1\left([0,1], \mathbb{R}^2\right) : \mathfrak{v}(0) = x_1, \mathfrak{v}(1) = x_2, |\mathfrak{v}(t)| \le \bar{R} \text{ and}& \\ \mathfrak{v}(t) \neq c_j \text{ for every } t \in [0,1], \text{ for every } j \in \{1, \dots, N\},& \\ \mathfrak{v} \text{ separates the centres according to the partition } P_j \big\};&\end{aligned}$$

namely in the correspondence (1.1)$\leftrightsquigarrow$(1.4) there holds

$$\widehat{\mathfrak{K}}_{P_j}^{x_1 x_2}([0,1]) \leftrightsquigarrow \widehat{K}_{P_j}^{x_1 x_2}([0,1]) \quad \mathfrak{K}_{P_j}^{x_1 x_2}([0,1]) \leftrightsquigarrow K_{P_j}^{x_1 x_2}([0,1]).$$

In what follows *we consider $h \in (\bar{h}, 0)$ and fixed*. Hence we will omit the dependence on h for the pieces of solutions of equation (1.1), to ease the notation. As we stated in Corollary 1.3, Theorem 1.1 enables us to characterized the dynamical system of the N-centre problem restricted on the energy shell

$$\mathcal{U}_h = \left\{ (x,v) \in \mathbb{R}^2 \setminus \{c_1, \dots, c_N\} \times \mathbb{R}^2 : \frac{1}{2}|v|^2 - V(x) = h \right\}$$

with a symbolic dynamics, where the symbols are the element of $\mathcal{P}$. Let us rewrite the Hamilton's equations

$$\begin{cases} \dot{x}(t) = v(t) \\ \dot{v}(t) = \nabla V(x(t)). \end{cases} \tag{6.1}$$

Such a system defines the vector field

$$\begin{aligned} X :& \mathbb{R}^2 \setminus \{c_1, \dots, c_N\} \times \mathbb{R}^2 \to T(\mathbb{R}^2 \setminus \{c_1, \dots, c_N\}) \times \mathbb{R}^2 \\ & (x,v) \mapsto (v, \nabla V(x)), \end{aligned}$$

which in turns generates the flow

$$\begin{aligned} \varphi^t :& \mathbb{R}^2 \setminus \{c_1, \dots, c_N\} \times \mathbb{R}^2 \to \mathbb{R}^2 \setminus \{c_1, \dots, c_N\} \times \mathbb{R}^2 \\ & (x_0, v_0) \mapsto (x(t; x_0, v_0), v(t; x_0, v_0)). \end{aligned}$$

It associates with (x_0, v_0) the solution of (6.1) having initial value $(x(0) = x_0, v(0) = p_0)$ evaluated at time t, and it is well defined for t in an open neighbourhood of 0. In general the flow is not complete (i.e. given (x_0, v_0) the solution $(x(t; x_0, v_0), v(t; x_0, v_0))$ is not defined for every $t \in \mathbb{R}$), due to the collisions, but we can complete it with the agreement that if there exists $t_* \in \mathbb{R}$ such that $x(\cdot\,; x_0, v_0)$ has a collision at t_*, then we extend the corresponding solution as an ejection-collision solution:

$$\varphi^{t_* + t}(x_0, v_0) := \varphi^{t_* - t}(x_0, v_0) \quad \forall t \in \mathbb{R}.$$

This implies in particular that at most two collisions occurs for every $(x,v) \in \mathbb{R}^2 \setminus \{c_1, \dots, c_N\} \times \mathbb{R}^2$, up to periodicity.

Furthermore the resulting flow is the same given by the Levi-Civita regularization (see Remark 4.43); hence φ^t *is continuous for every* t.

The energy shell $\mathcal{U}_h$ is a 3-dimensional submanifold of $\mathbb{R}^2 \setminus \{c_1, \dots, c_N\} \times \mathbb{R}^2$, which is invariant for X; hence it makes sense to consider the restriction $X_h := X|_{\mathcal{U}_h}$, for every $h \in (\bar{h}, 0)$. We consider the 2-dimensional submanifolds

$$\mathcal{U}^{\pm}_{h,\bar{R}} := \left\{(x,v) \in \mathcal{U}^h : |x| = \bar{R} \text{ and } \langle v, x\rangle \gtrless 0\right\}$$

which are some sort of cylinders in $\mathbb{R}^4$; thinking at (x,v) as a pair position-velocity, $\mathcal{U}^+_{h,\bar{R}}$ (respectively $\mathcal{U}^-_{h,\bar{R}}$) is the set of couples with position $x \in \partial B_{\bar{R}}(0)$, and velocity which points towards the external of (resp. towards the inner of) the ball $B_{\bar{R}}(0)$ and is not tangent to $\partial B_{\bar{R}}(0)$. For a point $(x,v) \in \mathcal{U}^+_{h,\bar{R}}$, the normal field to $\mathcal{U}^+_{h,\bar{R}}$ is

$$\mathcal{N}_{h,\bar{R}}(x,v) = \left(\frac{x}{\bar{R}}, 0\right).$$

The vector field X_h is transverse to $\mathcal{U}^+_{h,\bar{R}}$, in the sense that for every $(x,v) \in \mathcal{U}^+_{h,\bar{R}}$

$$\left\langle X_h(x,v), N_{h,\bar{R}}(x,v)\right\rangle = \frac{\langle x, v\rangle}{\bar{R}} > 0.$$

For every $(x,v) \in \mathcal{U}^+_{h,\bar{R}}$ we can define

$$\mathfrak{T}^{\pm}(x,v) := \left\{t \in (0,+\infty) : \varphi^t(x,v) \in \mathcal{U}^{\pm}_{h,\bar{R}}\right\}$$

which in general can be empty. Let us term

$$\left(\mathcal{U}^+_{h,\bar{R}}\right)^{\pm} := \left\{(x,v) \in \mathcal{U}^+_{h,\bar{R}} : \mathfrak{T}^{\pm}(x,v) \neq \emptyset\right\}.$$

There are points $(x,v) \in \left(\mathcal{U}^+_{h,\bar{R}}\right)^{\pm}$, since the periodic solutions we found in Theorem 1.1 do cross the circle $\{|x| = \bar{R}\}$ with velocity $\dot{x}$ satisfying the transversality condition $\langle x, \dot{x}\rangle \gtrless 0$ an infinite number of times. The continuous dependence of the solution on initial data and the transversality of $\mathcal{U}^+_{h,\bar{R}}$ with respect to X_h implies that $\left(\mathcal{U}^+_{h,\bar{R}}\right)^+$ is open in $\mathcal{U}_{h,\bar{R}}$.

We point out that, in order to fulfil the transversality condition, if $(x,v) \in \left(\mathcal{U}^+_{h,\bar{R}}\right)^+$ then its trajectory could pass trough the cylinder $\{|x| < \bar{R}\}$ (hence (x,v) could stay in $\left(\mathcal{U}^+_{h,\bar{R}}\right)^-$). Therefore, for $(x,v) \in \left(\mathcal{U}^+_{h,\bar{R}}\right)^+$, it makes sense to set

$$T^{\pm}_{\min} := \inf \mathfrak{T}^{\pm}(x,v).$$

For every $(x,v) \in \left(\mathcal{U}^+_{h,\bar{R}}\right)^+$ such that $T^-_{\min} < T^+_{\min}$, we consider $\{\varphi^t(x,v)\}_{t\in[T^-_{\min},T^+_{\min}]}$, i.e. the restriction of the trajectory starting from (x,v) to the first time interval needed to cross $B_{\bar{R}}(0)$. We define

$$\mathcal{U}^{\mathcal{P}}_{h,\bar{R}} := \Big\{(x,v) \in \left(\mathcal{U}^+_{h,\bar{R}}\right)^+ : T^-_{\min} < T^+_{\min},\ \{\varphi^t(x,v)\}_{t\in[T^-_{\min},T^+_{\min}]} \text{ parametrizes a self-intersections-free minimizer of } L_h \text{ in } \mathfrak{K}^{x(T^-_{\min})x(T^+_{\min})}_{P_j}, \text{ for some } P_j \in \mathcal{P}\Big\}.$$

It is non-empty, since our periodic orbits provide an infinite number of points satisfying these conditions. It is possible to define a *first return map* on $\mathcal{U}^{\mathcal{P}}_{h,\bar{R}}$ as

$$\mathcal{R}(x,v) := \varphi^{T^+_{\min}}(x,v).$$

We can also introduce an application $\chi : \mathcal{U}^{\mathcal{P}}_{h,\bar{R}} \to \mathcal{P}$ given by

$$\chi(x,v) := \begin{cases} P_j & \text{if } \{\varphi^t(x,v)\}_{t\in[T^-_{\min},T^+_{\min}]} \text{ parametrizes a path which} \\ & \ \text{separates the centres according to } P_j, \text{ with } P_j \in \mathcal{P} \\ Q_j & \text{if } \{\varphi^t(x,v)\}_{t\in[T^-_{\min},T^+_{\min}]} \text{ parametrizes} \\ & \text{an ejection-collision path, which collides in } c_j. \end{cases}$$

Finally, let us term

$$\Pi_h := \bigcap_{j\in\mathbb{Z}} \mathcal{R}^j(\mathcal{U}^{\mathcal{P}}_{h,\bar{R}}),$$

the set of initial data such that the corresponding solutions cross the circle $\partial B_{\bar R}(0)$ with velocity directed towards the exterior of the ball $B_{\bar R}(0)$ an infinite number of time in the future and in the past. Again, the periodic solutions found in Theorem 1.1 provide an infinite number of points in Π_h. Now, for every $(x,v) \in \Pi_h$, we set $\pi : \Pi_h \to \mathcal{P}^{\mathbb{Z}}$ as

$$\pi(x,v) = (P_{j_k})_{k\in\mathbb{Z}} \quad \text{where} \quad P_{j_k} := \chi(\mathcal{R}^{k-1}(x,p)).$$

Introduced the restriction $\mathfrak{R} := \mathcal{R}|_{\Pi}$, the proof of Corollary 1.3 reduces to the following Proposition.

Proposition 6.1. *Under the assumption of Theorem 1.1, the map π is continuous and surjective, and the diagram*

$$\begin{tikzcd} \Pi_h \arrow[r, "\mathfrak{R}"] \arrow[d, "\pi"] & \Pi_h \arrow[d, "\pi"] \\ \mathcal{P}^{\mathbb{Z}} \arrow[r, "T_r"] & \mathcal{P}^{\mathbb{Z}}, \end{tikzcd}$$

commutes.

We need some preliminary results. The first step is to obtain uniform bounds, below and above, for the time interval of the pieces of solution found in sections 3 and 4.

Lemma 6.2. *There exist $C_1, C_2 > 0$ such that for every $(x_0,x_1) \in (\partial B_{\bar R}(0))^2$ such that $|x_1 - x_0| < \bar\delta$, and for every $(x_2,x_3) \in (\partial B_R(0))^2$, for every $P_j \in \mathcal{P}$, there holds*

$$C_1 \le T_{ext}(x_0,x_1) \le C_2$$
$$C_1 \le T_{P_j}(x_2,x_3) \le C_2.$$

Proof. It is a straightforward consequence of Lemmas 5.1 and 5.2, and of Proposition 2.1. □

It will be useful to prove that, for a sequence of minimizers of M_h which separate the centres according to the same partition P_j, the convergence of the ends to $(\bar x_1, \bar x_2)$ is sufficient for the weak convergence in H^1 of the minimizers themselves; the limit path turns out to be minimal for M_h in $\mathfrak{K}_{P_j}^{\bar x_1 \bar x_2}([0,1])$

Lemma 6.3. *Let $(x_1^n, x_2^n) \subset (\partial B_{\bar R}(0))^2$ such that $(x_1^n,x_2^n) \to (\bar x_1, \bar x_2)$, let $P_j \in \mathcal{P}$; let $\mathfrak{u}_n$ be a local minimizers of M_h in $\mathfrak{K}_{P_j}^{x_1^n x_2^n}([0,1])$.*
Then there exists a subsequence $(\mathfrak{u}_{n_k})$ of $(\mathfrak{u}_n)$ and a minimizer $\bar{\mathfrak{u}} \in \mathfrak{K}_{P_j}^{\bar x_1 \bar x_2}([0,1])$ of M_h such that $\mathfrak{u}_{n_k} \rightharpoonup \bar{\mathfrak{u}}$ in H^1.

Proof. In order to prove that, up to subsequence, $(\mathfrak{u}_n)$ is weakly convergent, it is sufficient to show that $(\mathfrak{u}_n)$ is bounded in H^1. We know that

$$\|\mathfrak{u}_n\|_2^2 \le \bar R^2 \qquad \forall n,$$

hence it remains to check that there exists $C > 0$ such that

$$\|\dot{\mathfrak{u}}_n\|_2^2 \le C \qquad \forall n.$$

In the previous proof we saw that the minimality of $\mathfrak{u}_n$ implies that this inequality is satisfied with $C = C_8$ (see (5.4)). Now let us prove that the limit $\mathfrak{u}$ is a minimizer of M_h. We recall that in step 1) of the proof of Theorem 5.3, we proved the continuity of the function which associate to each couple of ends $p_1, p_2 \in \partial B_R(0)$ the length L (in the "normalized" problem) of the minimizers $u_{P_j}(\cdot; p_1, p_2; \epsilon)$, for every P_j. It is straightforward to check that the same property holds true for L_h with $h \neq -1$, and $x_1, x_2 \in \partial B_{\bar R}(0)$. Assume by contradiction that $\mathfrak{u}$ were not a local minimizer of M_h; by Proposition 4.7 it follows that $\mathfrak{u}$ cannot be a minimizer also of L_h, then there exists a path $\mathfrak{v} \in L_{P_j}^{\bar x_1 \bar x_2}([0,1])$ such that $L_h(\mathfrak{v}) \le L_h(\mathfrak{u})$. Let $\sigma_{\bar R}(\cdot\,; x_*, x_{**})$ be the shorter (in the Euclidean metric) arc of $\partial B_{\bar R}(0)$ connecting x_* with x_{**}, parametrized with constant velocity. As $|x_* - x_{**}|$ tends to zero, the length of $\sigma_{\bar R}$ tends to 0; hence there exists $n_0 \in \mathbb{N}$ such that

$$\widehat{\mathfrak{v}}_n(t) := \begin{cases} \sigma_{\bar R}(t; x_1^n, \bar x_1) & t \in [0,1/3] \\ \mathfrak{v}(3t-1) & t \in (1/3, 2/3] \\ \sigma_{\bar R}(t; \bar x_2, x_2^n) & t \in (2/3, 1], \end{cases}$$

is a path of $\mathfrak{K}_{P_j}^{x_1^n x_2^n}([0,1])$ such that $L_h(\mathfrak{v}_n) < L_h(\mathfrak{u}_n)$, a contradiction with the minimality of $\mathfrak{u}_n$. □

Proof of Proposition 6.1. **Step 1)** We start with surjectivity. Let $(P_{j_n})_{n\in\mathbb{Z}} \subset \mathcal{P}^{\mathbb{Z}}$. We can consider the finite sequences

$$(P_{j_0}), \quad (P_{j_{-1}}, P_{j_0}, P_{j_1}), \quad \dots \quad (P_{j_{-n}}, \dots, P_{j_{-1}}, P_{j_0}, P_{j_1}, \dots, P_{j_n}), \quad \dots.$$

To each sequence we associate the corresponding periodic solution of equation (1.1) with energy h given by Theorem 1.1, according to the notation

$$(P_{j_{-n}}, \dots, P_{j_{-1}}, P_{j_0}, P_{j_1}, \dots, P_{j_n}) \leftrightsquigarrow x^n(\cdot).$$

Up to time translations, we can take initial data $(x^n(0), \dot{x}^n(0)) \in \Pi_h$, in such a way that the first partition (or collision) determined by the solution $x^n(\cdot)$ is P_{j_0}, for every n.
The path parametrized by $x^n(\cdot)$ detects a sequence of points $(x^n_k)_{k\in\mathbb{Z}}$ of $\partial B_{\bar{R}}(0)$ given by the intersections of the trajectories in $\mathbb{R}^2$ with the circle itself, taken in the temporal order (of course, since $x^n(\cdot)$ is periodic, the sequence will be periodic, too).
We get a sequence of sequences:

$$(x^n_k)_{n\in\mathbb{N}} \quad \forall k \in \mathbb{Z}.$$

Now, $(x^n_0)_n$ stays in the compact set $\partial B_{\bar{R}}(0)$, therefore we can extract a subsequence $(x^{n_0}_0)_{n_0}$ which converges to $\bar{x}_0$. Analogously, $(x^{n_0}_1)_{n_0}$ stays in $\partial B_{\bar{R}}(0)$, therefore we can extract a subsequence $(x^{n_1}_1)_{n_1}$ which converges to $\bar{x}_1$. Proceeding in this way, for every $k \in \mathbb{Z}$ we have a sequence $(x^{n_k}_k)_{n_k}$ which converges to $\bar{x}_k$. Then we relabel as $(x^n_k)_n$ the diagonal sequence, namely $(x^{n_n}_k)_n$. It results

$$\lim_{n\to\infty} x^n_k = \bar{x}_k \qquad \forall k \in \mathbb{Z}. \tag{6.2}$$

For every $k \in \mathbb{Z}$, we connect the points $\bar{x}_{2k}$, $\bar{x}_{2k+1}$ with the unique external solution of (1.1) given by Theorem 3.1. Analogously, we connect $\bar{x}_{2k+1}$ and $\bar{x}_{2k+2}$ with the inner solution given by Theorem 4.12. We point out that a collision can occur just if $\alpha = 1$ and $\bar{x}_{2k+1} = \bar{x}_{2k+2}$. We can juxtapose these paths in a continuous manner, following the same gluing procedure already carried on in section 5 to define $\gamma_{(p_0,\dots,p_{2n})}$; in this way we obtain a continuous function $\bar{x}(\cdot) : \mathbb{R} \to \mathbb{R}^2$. We claim that it is a solution of (1.1) (in case $\alpha = 1$, it can be an ejection-collision solution) such that $(\bar{x}(0), \dot{\bar{x}}(0)) \in \Pi_h$ and $\pi((\bar{x}_0, \dot{\bar{x}}(0))) = (P_{j_k})_k$. The first step is to prove that, up to a subsequence, $(x^n(\cdot))$ converges to $\bar{x}(\cdot)$ uniformly on every compact set of $\mathbb{R}$. If $[a, b] \subset \mathbb{R}$ such that $\bar{x}(a) = \bar{x}_{2k}$ and $\bar{x}(b) = \bar{x}_{2k+1}$, with $k \in \mathbb{Z}$, then the uniform convergence in $[a, b]$ is a straightforward consequence of the continuous dependence of the external solutions by the end points (Theorem 3.1). On the other hand, if $[c, d] \subset \mathbb{R}$ with $\bar{x}(c) = \bar{x}_{2k+1}$ and $\bar{x}(d) = \bar{x}_{2k+2}$, then the uniform convergence has been proved in Lemma 6.3. From this, it is easy to obtain the uniform convergence for every compact subset of $\mathbb{R}$. Let us observe that since $\bar{x}|_{[c,d]}$ is a uniform limit of minimizers of L_h (and hence, up to reparametrizations, also of M_h), if $\bar{x}|_{[c,d]}$ has a collision, necessarily $\bar{x}|_{[c,d]}$ parametrizes an ejection-collision path (see Remark 4.42). Now, assume first that $\bar{x}(\cdot)$ has no collisions in $\mathbb{R}$. Let $[a, b] \subset \mathbb{R}$ be compact. In this case there exists $\bar{n} \in \mathbb{N}$ such that $x^n(\cdot)$ is collisions-free in $[a, b]$, as well. The function $V(\bar{x}(\cdot))$ is well defined in $\mathbb{R}$, and by regularity

$$\lim_{n\to\infty} \ddot{x}^n(t) = \lim_{n\to\infty} \nabla V(x^n(t)) = \nabla V(\bar{x}(t)), \tag{6.3}$$

with uniform convergence in $[a, b]$. Also, the derivative of $x^n(\cdot)$ is uniformly bounded in $[a, b]$ for the conservation of the energy:

$$|\dot{x}^n(t)| = \sqrt{2(V(x^n(t)) + h)} \leq \sqrt{2\,(C + h)} \qquad \forall t \in [a, b], \forall n \geq \bar{n}.$$

Hence, up to subsequence, there exists a point $\bar{t} \in (a, b)$ such that $\dot{x}_n(t)$ is convergent in $\mathbb{R}^2$. This fact, together with (6.3), implies that $(\dot{x}^n(\cdot))$ converges in $\mathcal{C}^1([a, b])$, and hence $(x^n(\cdot))$ converges in the $\mathcal{C}^2([a, b])$ to $\bar{x}(\cdot)$, for every compact subset $[a, b] \in \mathbb{R}$. This means that $\bar{x}$ is a $\mathcal{C}^2$ solution of (1.1) with energy h on $[a, b]$ and this argument works in every compact subset of $\mathbb{R}$. We point out that the uniform convergence is sufficient to say that, in its k-th passage inside $B_{\bar{R}}(0)$, $\bar{x}(\cdot)$ separates the centres according to P_{j_k}, namely $\pi(\bar{x}(0), \dot{\bar{x}}(0))) = (P_{j_k})_{k\in\mathbb{Z}}$.
Now, we are left to examine what happens if a collision occurs. Let

$$T_c(\bar{x}) := \{t \in \mathbb{R} : \bar{x}(t) = c_j \text{ for some } j \in \{1, \dots, N\}\}.$$

The $\mathcal{C}^2$-convergence of $(x^n(\cdot))$ to $\bar{x}(\cdot)$ is still true in every compact subset of $\mathbb{R} \setminus T_c(\bar{x})$, hence we obtain an ejection-collision solution of (1.1) with energy h and $\pi(\bar{x}_0, \dot{\bar{x}}(0))) = (P_{j_k})_{k\in\mathbb{Z}}$.
We point out that this is possible just for $\alpha = 1$ and $(P_{j_k}) \in \mathcal{P}^{\mathbb{Z}}$ such that

- (P_{j_k}) is periodic and satisfies the conditions of points (*ii-b*) or (*ii-c*) of Theorem 1.1.

- up to a finite number of applications of the right shift, $P_{j_0} \in \mathcal{P}_1$ and the sequence is symmetric, i.e. $P_{j_{-n}} = P_{j_n}$ for every n.

Step 2) It remains to show that π is continuous. Let $(x_0, v_0) \in \Pi_h$. We would like to prove that given $\lambda > 0$ there exists $\varrho > 0$ such that for every $(x, v) \in \Pi_h$:

$$|(x,v) - (x_0, v_0)| < \varrho \Rightarrow \sum_{m \in \mathbb{Z}} \frac{d_1(\pi_m(x,v), \pi_m(x_0,v_0))}{2^{|m|}} < \lambda,$$

where π_m is the projection $\pi_m : \Pi_h \to \mathcal{P}$ defined by

$$\pi_m(x,v) := \chi(\mathfrak{R}^{m-1}(x,v)),$$

i.e. π_m associate to (x,v) the partition that the corresponding solution induces in its m-th passage inside $B_{\bar{R}}(0)$. Let us observe that there exists $m_0 \in \mathbb{N}$ such that

$$\sum_{|m| > m_0} \frac{1}{2^{|m|}} < \lambda.$$

Hence it is sufficient to show that, if we take two initial data sufficiently close, then the corresponding solutions induce the same partitions P_{j_k} of the centres, for $k \in \{-m_0, \dots, m_0\}$.

Thanks to Lemmas 6.2 and 5.2, we can fix a time interval $[-a, a]$ such that each solution with initial data in Π_h passes at least $2m_0 + 1$-times inside $B_{\bar{R}}(0)$ in $[-a, a]$. If the solution of (1.1) with starting point (x_0, v_0) is collisions-free, then there exists $\mu > 0$ such that

$$|x(t; x_0, v_0) - c_j| \geq \mu \qquad \forall t \in [-a, a].$$

If (x, v) is sufficiently close to (x_0, v_0), then the continuous dependence applies:

$$\forall \lambda \in \left(0, \frac{\mu}{2}\right) \ \exists \varrho > 0 : |(x,v) - (x_0,v_0)| < \varrho \Rightarrow |x(t; x, v) - x(t; x_0, v_0)| < \lambda.$$

This implies that $x(\cdot\,; x, v)$ is collisions-free and detects the same partitions of $x(\cdot; x_0, v_0)$ in $[-a, a]$. In particular, $\pi_m(x, v) = \pi_m(x, v)$ for every $m \in \{-m_0, \dots, m_0\}$. This proves the continuity for non-collision initial data. But nothing change if we consider $(x_0, v_0) \in \Pi_h$ such that $x(\cdot; x_0, v_0)$ has a collision: indeed we introduced a regularization trough the Levi-Civita transform (see Remark 4.43 on the Levi-Civita transform), so that the continuous dependence applies also in this case. $\square$

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Dipartimento di Matematica e Applicazioni, Università degli Studi di Milano-Bicocca, Via Bicocca degli Arcimboldi, 8, 20126 Milano, Italy
E-mail address: n.soave@campus.unimib.it
E-mail address: susanna.terracini@unimib.it